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A Note on Generalized DDI Graph of a Vector Space

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Abstract

In 2016, Angsuman Das introduced the Subspace Inclusion Graph of a Vector Space $In(\mathbb{V})$ [1]. It is a graph $In(\mathbb{V}) = (V, E)$ with V as the collection of nontrivial proper subspaces of $\mathbb{V}$ and $W_1, W_2 \in V$ are adjacent if either $W_1 \subset W_2$ or $W_2 \subset W_1$. Also we studied about the DDI Graph of a Vector Space. It is a graph $\Gamma_{\mathrm{DDI}}(\mathbb{V})$ with the vertex set as the collection of non-trivial proper subspaces of a vector space $\mathbb{V}$ and two vertices W_1& W_2 are adjacent if and only if $(\dim(W_2) - \dim(W_1)) \in [\dim(W_1), \dim(W_2)]$ [without of loss generality we assume that $\dim(W_2) \geq \dim(W_1)$]. In this paper, we generalize the definition DDI graph of a Vector Space. Let $\mathbb{V}$ be a finite dimensional vector space and let S_i be the set of all proper subspaces of dimension i. Then Generalized DDI graph $\Gamma_{\mathrm{GDDI}}(\mathbb{V})$ of a vector space $\mathbb{V}$ is a graph with the vertex set, $V(\Gamma_{GDDI}(\mathbb{V}) = \{S_1, S_2, \dots, S_{n-1}\}$ and two vertices S_i & S_j are adjacent if and only if $(j - i) \in [i, j]$ [without of loss generality we assume that $j \geq i$]. We investigate the structure and graph theoretical properties like connectivity, hamiltonicity, independence and covering number etc for Generalized DDI Graph of a Vector Space.

Keywords - Connectivity, Hamiltonian, Independence number, covering number.

1 Introduction

The study of algebraic structures, using the properties of graphs, has become an exciting research topic in the last three decades, leading to many fascinating results and questions. There are many papers assigning a graph to a ring or group and investigating algebraic properties using the associated graph. In this paper, we assign a graph to a finite dimensional vector space $\mathbb{V}$ and investigate algebraic properties of the vector space using graph theoretical concepts.

We consider simple graphs which are undirected, with no loops or multiple edges. For any graph $\Gamma(\mathbb{V})$, we denote the sets of the vertices and edges of $\Gamma(\mathbb{V})$ by $V(\Gamma(\mathbb{V}))$ and $E(\Gamma(\mathbb{V}))$, respectively. A graph G is said to be **complete** if every pair of vertices are adjacent and a complete graph on n vertices is denoted by K_n. A graph G is said to be **bipartite** if the vertex set is partitioned into two subsets V_1 and V_2 such that every edge of G joins a vertex of V_1 and a vertex of V_2. A **complete bipartite graph** is the bipartite graph in which all possible edges are included and if $|V_1| = m$ and $|V_2| = n$ then it is denoted by $K_{m,n}$. A graph is said to be **triangulated** if for any vertex u in $V(G)$, there exist v, w in $V(G)$, such that (u, v, w) is a triangle. A **clique** in a graph G is a complete subgraph of G. For a real number x, the **floor** $\lfloor x \rfloor$ of x is the greatest integer not exceeding x. The **ceiling** $\lceil x \rceil$ of x is the smallest integer not less than x. For terminology in graph theory we refer Chatrand and West [3,4].

We use the following theorems.

Theorem 1.1. Let G be a connected graph. If G is a Hamiltonian graph, then for every nonempty proper subset S of vertices of G, the number of connected components of $G \backslash S$ is less than or equal to the cardinality of S.

Theorem 1.2. A graph G is Euelrian if and only if degree of every vertex is even.

2 Main Results

Definition 2.1. Let $\mathbb{V}$ be a finite dimensional vector space and let S_i be the set of all proper subspaces of dimension i. Then generalized DDI graph $\Gamma_{\mathrm{GDDI}}(\mathbb{V})$ of a vector space $\mathbb{V}$ is a graph with the vertex set, $V(\Gamma_{GDDI}(\mathbb{V})) = \{S_1, S_2, \ldots, S_{n-1}\}$ and two vertices S_i & S_j are adjacent iff $(j - i) \in [i, j]$ without loss of generality we assume that $j \geq i$.

Lemma 2.2. Let $\mathbb{V}$ be a finite dimensional vector space. Then the following can be observed about the generalized DDI graph $\Gamma_{GDDI}(\mathbb{V})$ of $\mathbb{V}$.

(1) If $\dim(\mathbb{V}) \geq 3$, then $\Gamma_{GDDI}(\mathbb{V})$ is connected.

(2) If $\dim(\mathbb{V}) \geq 4$, then $\Gamma_{GDDI}(\mathbb{V})$ is not complete.

(3) If $\mathbb{V}$ is a finite dimensional vector space and W is a subspace of $\mathbb{V}$ with dimension greater than 1, then $\Gamma_{\mathrm{G}}(W)$ is a subgraph of $\Gamma_{GDDI}(\mathbb{V})$.

Lemma 2.3. Let S_i and S_j be two distinct vertices of a generalized DDI graph. Then S_i is adjacent to S_j if and only if $j \leq \left\lfloor \frac{i}{2} \right\rfloor$ or $j \geq 2i$.

Proof: Let S_i and S_j be two distinct vertices of a generalized DDI graph.

Case: (i) Let $j \leq \left\lfloor \frac{i}{2} \right\rfloor$, then $i - j \leq \left\lceil \frac{i}{2} \right\rceil \in \left[\left\lfloor \frac{i}{2} \right\rfloor, i\right]$. Hence S_i is adjacent to S_j.

Case: (ii) Let $j \geq 2i$, then $j - i \geq\ i \in [i, 2i]$.\$ Hence S_i is adjacent to S_j.

Conversely let us assume that S_i is adjacent to S_j. By definition of $\Gamma_G(\mathbb{V})$, $(j - i) \in [i, j]$ or $(i - j) \in [j, i]$. If $(j - i) \in [i, j]$, then $j \geq 2i$. If $(i, j) \in [j, i]$, then $j < \frac{i}{2}$.\$ Since j is an integer, $j \leq \left\lfloor \frac{i}{2} \right\rfloor$.

Hence the theorem.

Theorem 2.4. Let $\mathbb{V}$ be a n-dimensional vector space where $n \geq 2$ and let S_i be a vertex of generalized DDI graph $\Gamma_{GDDI}(\mathbb{V})$. Then degree of S_i, $\deg(S_i) = \begin{cases} n-2 & \text{if } i = 1 \\ n-3 & \text{if } i = 2 \\ \left\lfloor \frac{i}{2} \right\rfloor & \text{if } i \geq 3 \text{ and } i+1 \leq n \leq 2i \\ n - \left\lceil \frac{3i}{2} \right\rceil & \text{if } i \geq 3 \text{ and } n > 2i \end{cases}$

Proof: Let $\mathbb{V}$ be a n-dimensional vector space where $n \geq\ 2$ and let S_i be a vertex of generalized DDI graph Γ_{GDDI} $(\mathbb{V})$. From the definition of $\Gamma_{GDDI}(V)$, the vertex S_1 is adjacent to all S_i for $2 \leq\ i \leq\ n-1$. Hence $\deg(S_1) = n - 2$. Also the vertex S_2 is not adjacent to only S_3 and so $\deg(S_2) = n - 3$.

Case (i) : If $i \geq\ 3$ and $i + 1 \leq\ n \leq\ 2i$, then by Lemma 2.4, S_i is adjacent to $\left\{1, 2, \dots, \left\lfloor \frac{i}{2} \right\rfloor\right\}$ and so $\deg(S_i) = \left\lfloor \frac{i}{2} \right\rfloor$.

Case (ii) : If $i \geq 3$ and $n > 2i$, then Lemma 2.3, S_i is not adjacent to $\left\{\left\lfloor \frac{i}{2} \right\rfloor + 1, \left\lfloor \frac{i}{2} \right\rfloor + 2, \dots, 2i - 1\right\}$. Hence $\deg(S_i) =\ n - 1 - \left\lfloor \frac{i-1}{2} \right\rfloor - i =\ n - 1 - i - \left(\left\lceil \frac{i}{2} \right\rceil\ - 1\right) =\ n - i - \left\lceil \frac{i}{2} \right\rceil = n - \left\lceil \frac{3i}{2} \right\rceil$.

Theorem 2.5. The generalized DDI graph $\Gamma_{GDDI}(\mathbb{V})$ is regular if and only if $\dim(\mathbb{V}) \leq 3$.

Proof: If $\dim(\mathbb{V}) \leq 3$, then $\Gamma_{GDDI}(\mathbb{V})$ is either trivial or $\Gamma_{GDDI}(\mathbb{V}) \cong K_2$ and so $\Gamma_{GDDI}(\mathbb{V})$ is regular. Conversely assume that $\Gamma_{GDDI}(\mathbb{V})$ is regular. If $\dim(\mathbb{V}) = n > 3$, then degree of S_1 and S_2 are $n - 2$ and $n - 3$ respectively and so $\Gamma_{\mathrm{GDDI}}(\mathbb{V})$ is not regular which is a contradiction. Hence $\dim(\mathbb{V}) \leq 3$.

Theorem 2.5. The generalized DDI graph $\Gamma_{GDDI}(\mathbb{V})$ is complete bipartite if and only if $\dim(\mathbb{V}) = 3$ or 4.
Proof: If $\dim(\mathbb{V}) = 3$, then $\Gamma_{GDDI}(\mathbb{V}) \cong K_2$ and if $\dim(\mathbb{V}) = 4$, then $\Gamma_{GDDI}(\mathbb{V}) \cong K_{1,2}$ and so $\Gamma_{GDDI}(\mathbb{V})$ is complete bipartite. Conversely assume that $\Gamma_{GDDI}(\mathbb{V})$ is complete bipartite. Suppose $\dim(\mathbb{V}) \geq 5$, then the girth is 3 and so $\Gamma_{\mathrm{GDDI}}(\mathbb{V})$ cannot be complete bipartite which is a contradiction and so $\dim(\mathbb{V}) < 5$. If $\dim(\mathbb{V}) = 2$, then $\Gamma_{GDDI}(\mathbb{V})$ is trivial. Hence dim $(\mathbb{V})$ is either 3 or 4.

Theorem 2.7. The generalized DDI graph $\Gamma_{GDDI}(\mathbb{V})$ is not Hamiltonian.

Proof. Let $\Gamma_{GDDI}(\mathbb{V})$ be a generalized DDI graph of a n-dimensional vector space $\mathbb{V}$. Let $S = \left\{S_1, S_2, \dots, S_{\left\lfloor\frac{n-2}{2}\right\rfloor}\right\}$, then the removal of elements in S from $\Gamma_{GDDI}(\mathbb{V})$ results the disconnected graph with $\left\lceil\frac{n}{2}\right\rceil$ number of connected components. Hence the number of connected components in $\Gamma_{GDDI}(\mathbb{V})\backslash S$ is $\left\lceil\frac{n}{2}\right\rceil > \left\lfloor\frac{n-2}{2}\right\rfloor$. By Theorem 1.1, $\Gamma_{GDDI}(\mathbb{V})$ is not Hamiltonian.

Theorem 2.8. The generalized DDI graph $\Gamma_{GDDI}(\mathbb{V})$ cannot be Eulerian.

Proof. Let $\Gamma_{GDDI}(\mathbb{V})$ be a generalized DDI graph of a n-dimensional vector space $\mathbb{V}$. By Theorem 2.4, either the degree of S_1 or the degree of S_2 is odd and so by Theorem 1.2 $\Gamma_{GDDI}(\mathbb{V})$ is not Eulerian.

Theorem 2.9. Let $\Gamma_{GDDI}(\mathbb{V})$ be a generalized DDI graph of a $n-$dimensional vector space $\mathbb{V}$ where $n \geq 3$. Then the independence number, $\beta(\Gamma_{GDDI}(\mathbb{V})) = \left\lfloor\frac{n}{2}\right\rfloor$

Proof. Let $\Gamma_{GDDI}(\mathbb{V})$ be a generalized DDI graph of a $n-$dimensional vector space $\mathbb{V}$. Clearly the set $S = \left\{S_{\left\lceil\frac{n}{2}\right\rceil}, S_{\left\lceil\frac{n}{2}\right\rceil+1}, \dots, S_{n-1}\right\}$ is not adjacent pairwise and so S is independent set in $\Gamma_{GDDI}(\mathbb{V})$. Also S is an maximal since if there exists a S_i where $i < \left\lceil\frac{n}{2}\right\rceil$ which is adjacent to S_{n-1}. Hence S is a maximum independent set. The number of elements in S is $n - 1 - \left(\left\lceil\frac{n}{2}\right\rceil - 1\right) = \left\lfloor\frac{n}{2}\right\rfloor$. Hence $\beta\left(\Gamma_{GDDI}(\mathbb{V})\right) = \left\lfloor\frac{n}{2}\right\rfloor$.

Theorem 2.10. Let $\Gamma_{GDDI}(\mathbb{V})$ be a generalized DDI graph of a $n-$dimensional vector space $\mathbb{V}$ where $n \geq 3$. Then the covering number, $\alpha\left(\Gamma_{GDDI}(\mathbb{V})\right) = \left\lceil\frac{n}{2}\right\rceil - 1$.

Proof. Given $\dim(\mathbb{V}) \geq 3$. Since $\alpha\left(\Gamma_{\mathrm{GDDI}}(\mathbb{V})\right) + \beta\left(\Gamma_{\mathrm{GDDI}}\ (\mathbb{V})\right) = \dim(\mathbb{V}) - 1 = n - 1$,

$\beta\left(\Gamma_{\mathrm{GDDI}}(\mathbb{V})\right) = n - 1 - \alpha\left(\Gamma_{\mathrm{GDDI}}(\mathbb{V})\right) = n - 1 - \left\lfloor\frac{n}{2}\right\rfloor = \left\lceil\frac{n}{2}\right\rceil - 1.$

Theorem 2.11. Let $\Gamma_{GDDI}(\mathbb{V})$ be a generalized DDI graph of a n −dimensional vector space $\mathbb{V}$ where $n \geq 3$. Then the edge covering number, $\alpha_1\left(\Gamma_{GDDI}(\mathbb{V})\right) = \left\lfloor\frac{n}{2}\right\rfloor$.

Proof. If $\dim(\mathbb{V}) = n$ and is odd, clearly the set $\{e_1 = (S_1, S_{i+1}), e_2 = (S_2, S_{i+2}), \ldots, e_i = (S_i, S_{2i})\}$ where $i = \frac{n-1}{2}$ is a minimum edge cover and so $\alpha_1\left(\Gamma_{GDDI}(\mathbb{V})\right) = i = \frac{n-1}{2} = \left\lfloor\frac{n}{2}\right\rfloor$.
If $\dim(\mathbb{V}) = n$ and is even, clearly the set $\{e_1 = (S_1, S_{i+1}), e_2 = (S_2, S_{i+2}), \ldots, e_{i-1} = (S_{i-1}, S_{2i-1}), e_i = (S_i, S_1)\}$ where $i = \frac{n}{2}$ is a minimum edge cover and so $\alpha_1\left(\Gamma_{GDDI}(\mathbb{V})\right) = i = \frac{n}{2} = \left\lfloor\frac{n}{2}\right\rfloor$.

Theorem 2.12. Let $\Gamma_{GDDI}(\mathbb{V})$ be a generalized DDI graph of a n −dimensional vector space $\mathbb{V}$ where $n \geq 3$. Then the edge independence number, $\beta_1\left(\Gamma_{GDDI}(\mathbb{V})\right) = \left\lceil\frac{n}{2}\right\rceil - 1$.

Proof. Since $\dim(\mathbb{V}) \geq 3$ and $\alpha_1\left(\Gamma_{\mathrm{GDDI}}(\mathbb{V})\right) + \beta_1\left(\Gamma_{\mathrm{GDDI}}(\mathbb{V})\right) = \dim(\mathbb{V}) - 1 = n - 1$, $\beta_1\left(\Gamma_{\mathrm{GDDI}}(\mathbb{V})\right) = n - 1 - \alpha_1\left(\Gamma_{\mathrm{GDDI}}(\mathbb{V})\right) = n - 1 - \left\lfloor\frac{n}{2}\right\rfloor = \left\lceil\frac{n}{2}\right\rceil - 1.$

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