Review of International Geographical Education | RIGEO | 2020

RIGEO

ISSN: 2146 - 0353

Review of International GEOGRAPHICAL EDUCATION

www.rigeo.org

An Introduction to DDI Graph of a Vector Space

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Abstract

In 2016, Angsuman Das introduced the Subspace Inclusion Graph of a Vector Space $In(\mathbb{V})$ [2]. It is a graph $In(\mathbb{V}) = (V, E)$ with V as the collection of nontrivial proper subspaces of $\mathbb{V}$ and $W_1, W_2 \in V$ are adjacent if either $W_1 \subset W_2$ or $W_2 \subset W_1$. In this paper, we introduce a DDI graph of a Vector Space. Let $\mathbb{V}$ be a finite-dimensional vector space. Then DDI graph $\Gamma_{\text{DDI}}(\mathbb{V})$ of a vector space $\mathbb{V}$ is a graph with the vertex set as the collection of non-trivial proper subspaces of a vector space $\mathbb{V}$ and two vertices W_1& W_2 are adjacent if and only if $(\dim(W_2) - \dim(W_1)) \in [\dim(W_1), \dim(W_2)]$ [without of loss generality we assume that $\dim(W_2) \geq \dim(W_1)$]. We investigate the structure and graph theoretical properties like connectivity, hamiltonicity, diameter, girth, etc for the Dimension Difference Interval Graph of a Vector Space.

Keywords - Connectivity, Hamiltonian, Diameter, Girth.

1 Introduction

The study of algebraic structures using graph properties has become an exciting research topic in the last three decades, leading to many fascinating results and questions. Many papers assign a graph to a ring or group and investigate algebraic properties using the associated graph. In this paper, we assign a graph to a finite-dimensional vector space $\mathbb{V}$ and investigate algebraic properties of the vector space using graph theoretical concepts.

We consider simple undirected graphs with no loops or multiple edges. For any graph $\Gamma(\mathbb{V})$, we denote the sets of the vertices and edges of $\Gamma(\mathbb{V})$ by $V(\Gamma(\mathbb{V}))$ and $E(\Gamma(\mathbb{V}))$, respectively. A graph G is said to be **complete** if every pair of vertices are adjacent and a complete graph on n vertices is denoted by K_n. A graph G is said to be **bipartite** if the vertex set is partitioned into two subsets V_1 and V_2 such that every edge of G joins a vertex of V_1 and a vertex of V_2. A **complete bipartite graph** is the bipartite graph in which all possible edges are included and if $|V_1| = m$ and $|V_2| = n$ then it is denoted by $K_{m,n}$. If G has a $u - v$ path, then the distance from u to v, written as $d(u, v)$ is the least length of a $u - v$ path. If G has no such path, then $d(u, v) = \infty$. A graph is said to be **triangulated** if for any vertex u in $V(G)$, there exist v, w in $V(G)$, such that (u, v, w) is a triangle. A **clique** in a graph G is a complete subgraph of G. The order of the largest clique in a graph G is its **clique number**, which is denoted by $\omega(G)$. If a graph can be drawn in the plane without crossing edges then it is called a **planar graph**. For a real number x, the **floor** $\lfloor x \rfloor$ of x is the greatest integer not exceeding x. The **ceiling** $\lceil x \rceil$ of x is the smallest integer not less than x. For terminology in graph theory we refer Chatrand and West [3,4].

Throughout this paper, even if it is not mentioned explicitly, the underlying field is $\mathbb{F}$ and $\mathbb{V}$ is finite dimensional. We investigate the structure and graph theoretical properties like connectivity, hamiltonicity, diameter, girth etc for $\Gamma(\mathbb{V})$.

2 Main Results

Definition 2.1. Let $\mathbb{V}$ be a finite dimensional vector space. Then dimension difference interval graph $\Gamma_{\mathrm{DDI}}(\mathbb{V})$ of a vector space $\mathbb{V}$ is a graph with the vertex set as the collection of non-trivial proper subspaces of the vector space $\mathbb{V}$ and two vertices W_1 & W_2 are adjacent if and only if $(\dim(W_2) - \dim(W_1)) \in [\dim(W_1), \dim(W_2)]$. Without loss of generality we assume that $\dim(W_2) \geq \dim(W_1)$.

Lemma 2.2. If W_1 is a 1-dimensional subspace and W_2 is any subspace of $\mathbb{V}$ with $\dim(W_2) = m \geq 2$ then W_1 is adjacent to W_2 in $\Gamma_{\mathrm{DDI}}(\mathbb{V})$.

Proof: By the definition of $\Gamma_{\mathrm{DDI}}(\mathbb{V})$, $(\dim(W_2) - \dim(W_1)) = m - 1$. Clearly $m - 1 \in [1, m]$. Hence W_1 is adjacent to W_2.

Lemma 2.3. Let $\mathbb{V}$ be a finite dimensional vector space. Then the following can be observed about the DDI graph $\Gamma_{DDI}(\mathbb{V})$ of $\mathbb{V}$.

(i) Let S_1 be the set of all 1-dimensional subspaces and S_2 be the set of all proper subspaces of $\mathbb{V}$ with dimension greater than 1. Then every element of S_1 is adjacent to every element of S_2.

(ii) If $\dim(\mathbb{V}) \geq 3$, then $\Gamma_{\mathrm{DDI}}(\mathbb{V})$ is connected.

(iii) If $\dim(\mathbb{V}) \geq 3$, then the diameter, $\mathrm{diam}\left(\Gamma_{\mathrm{DDI}}(\mathbb{V})\right) = 2$.

Lemma 2.4. If W_1 & W_2 are two distinct proper subspaces of $\mathbb{V}$ of same dimension then W_1 is not adjacent to W_2.

Proof: Let W_1 & W_2 be two distinct m-dimensional proper subspaces of $\mathbb{V}$. Suppose W_1 is adjacent to W_2, then by definition of $\Gamma(\mathbb{V}), \dim(W_2) - \dim(W_1) = 0 \notin \{m\}$ which is a contradiction. Hence W_1 is not adjacent to W_2.

Lemma 2.5. Let W be a subspace of a finite dimensional vector space $\mathbb{V}$ with $\dim(W) = m > 1$ and let W_1 be any proper subspace of $\mathbb{V}$. Then W is adjacent to to W_1 iff $\dim(W_1) \leq \left\lfloor \frac{m}{2} \right\rfloor$ or $\dim(W_1) \geq 2m$.

Proof: Let W be a m −dimensional subspace of a n −dimensional vector space $\mathbb{V}$. Let W_1 be any another proper subspace of $\mathbb{V}$.

Case: (i) Let $\dim(W_1) \leq \left\lfloor \frac{m}{2} \right\rfloor$. Then $(\dim(W) - \dim(W_1)) \geq \left\lceil \frac{m}{2} \right\rceil \in \left[\left\lfloor \frac{m}{2} \right\rfloor, m\right]$.\$ Hence W is adjacent to W_1.

Case: (ii) Let $\dim(W_1) \geq 2m$. Then $(\dim(W_1) - \dim(W)) \geq m \in [m, 2m]$. Hence W is adjacent to W_1.

Conversely let us assume that W is adjacent to W_1. Let $\dim(W_1) = m_1$. By definition of $\Gamma_{\mathrm{DDI}}(\mathbb{V})$, $(m_1 - m) \in [m, m_1]$ or $(m - m_1) \in [m_1, m]$. If $(m_1 - m) \in [m, m_1]$ then $m_1 \geq 2m$. If $(m - m_1) \in [m_1, m]$ then $m_1 \leq \frac{m}{2}$. Since m_1 is an integer, $m_1 \leq \left\lfloor \frac{m}{2} \right\rfloor$.

Hence the theorem.

Lemma 2.6. If $\mathbb{V}$ is a finite dimensional vector space over a field $\mathbb{F}$ and W is a subspace of $\mathbb{V}$ with dimension greater than 1, then $\Gamma_{\mathrm{DDI}}(W)$ is a subgraph of $\Gamma_{\mathrm{DDI}}(\mathbb{V})$.

Proof: It follows from the definition of $\Gamma_{\mathrm{DDI}}(\mathbb{V})$ and the fact that every subspace of W is also a subspace of $\mathbb{V}$.

Lemma 2.7. If $\dim(\mathbb{V}) = 2$, then $\Gamma_{\mathrm{DDI}}(\mathbb{V})$ is a totally disconnected graph.

Proof: As $\dim(\mathbb{V}) = 2$, only non-trivial proper subspaces of $\mathbb{V}$ are of dimension 1. Clearly, no two vertices of $\Gamma_{\mathrm{DDI}}(\mathbb{V})$ are adajcent.

Lemma 2.8. For any vector space $\mathbb{V}$ with $\dim(\mathbb{V}) > 1$, $\Gamma_{\mathrm{DDI}}(\mathbb{V})$ can never be complete.

Proof: Since $\dim(\mathbb{V}) > 1$, there exists at least two linearly independent vectors α and β in $\mathbb{V}$. Then $W_1 = \langle\alpha\rangle$ and $W_2 = \langle\beta\rangle$ are two non-trivial proper subspaces of $\mathbb{V}$ which are not adjacent. So $\Gamma_{\mathrm{DDI}}(\mathbb{V})$ is not complete.

Lemma 2.9. If $\mathbb{V}$ is an n −dimensional vector space with $n \geq 3$, then $\Gamma_{\mathrm{DDI}}(\mathbb{V})$ is not planar.

Proof: Let $\mathbb{V}$ be a finite dimensional vector space of dimension at least 3. Then $\mathbb{V}$ has subspaces of dimension either 1 or 2. Also we know that the underlying field has at least two elements. Then the number of 1 −dimensional and 2 −dimensional subspaces are at least 7. Thus $K_{7,7}$ is a subgraph of $\Gamma_{\mathrm{DDI}}(\mathbb{V})$. Hence $\Gamma_{\mathrm{DDI}}(\mathbb{V})$ is not planar.

Lemma 2.10. If $\dim(\mathbb{V}) \geq 7$, then $\Gamma_{\mathrm{DDI}}(\mathbb{V})$ is triangulated.

Proof: Let $\mathbb{V}$ be a finite dimensional vector space of dimension at least 7. Let W be any proper subspace of $\mathbb{V}$. To prove that W lies on a triangle in $\Gamma_{\mathrm{DDI}}(\mathbb{V})$.

Case 1: $\dim(W) = 1$. Since $\dim(\mathbb{V}) \geq 7$, there exists two subspaces W_1 & W_2 of dimensions 2 and 4 respectively. Also $W - W_1 - W_2 - W$ form a triangle.

Case 2: $\dim(W) = 2$. Since $\dim(\mathbb{V}) \geq 7$, there exists two subspaces W_1 & W_2 of dimensions 1 and 4 respectively. Also $W - W_1 - W_2 - W$ form a triangle.

Case 3: $\dim(W) = 3$. Since $\dim(\mathbb{V}) \geq 7$, there exists two subspaces W_1 & W_2 of dimensions 1 and 6 respectively. Also $W - W_1 - W_2 - W$ form a triangle.

Case 4: $\dim(W) \geq 4$. Then there exists two subspaces W_1 & W_2 of dimensions 1 and 2 respectively such that $W - W_1 - W_2 - W$ form a triangle.

Theorem: 2.11. Let $\mathbb{V}$ be an n −dimensional vector space. Let m be the least positive integer such that $n \leq 2^m$. Then the clique number $\omega\big(\Gamma_{\mathrm{DDI}}(\mathbb{V})\big) = m$.

Proof: Let $\mathbb{V}$ be an n −dimensional vector space. Let m be the least positive integer such that $n \leq 2^m$.

Let the subspaces $W_1, W_2, \dots, W_m$ be of dimension $2^0, 2^1, \dots, 2^{m-1}$ respectively. We know that these are pairwise adjacent. Then $\omega\big(\Gamma_{\mathrm{DDI}}(\mathbb{V})\big) \geq m$.

Suppose there exist a subspace W_i of $\mathbb{V}$ such that $\dim(W_i) = k \neq 2^l$ where $l = 0,1,\dots,m-1$. (i.e) W_i is a subspace whose dimension is not a power of 2. Therefore $2^j < k < 2^{j+1}$ for some j. Then W_i is not adjacent to W_j for $j = 1,2,\dots,m$. Hence $\omega\big(\Gamma_{\mathrm{DDI}}(\mathbb{V})\big) = m$.

Theorem 2.12. Let $\mathbb{V}$ be an n −dimensional vector space. Let m be the least positive integer such that $n \leq 2^m$. Then the chromatic number $\chi\big(\Gamma_{\mathrm{DDI}}(\mathbb{V})\big) = m$.

Proof: By Theorem 2.11, $\chi\big(\Gamma_{\mathrm{DDI}}(\mathbb{V})\big) \geq m$. For any subspace W of $\mathbb{V}$, colour W with jth colour if $2^{j-1} \leq \dim(W) < 2^j$. Clearly no two adjaccent vertices get a same colour. Hence $\chi\big(\Gamma_{\mathrm{DDI}}(\mathbb{V})\big) = m$.

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