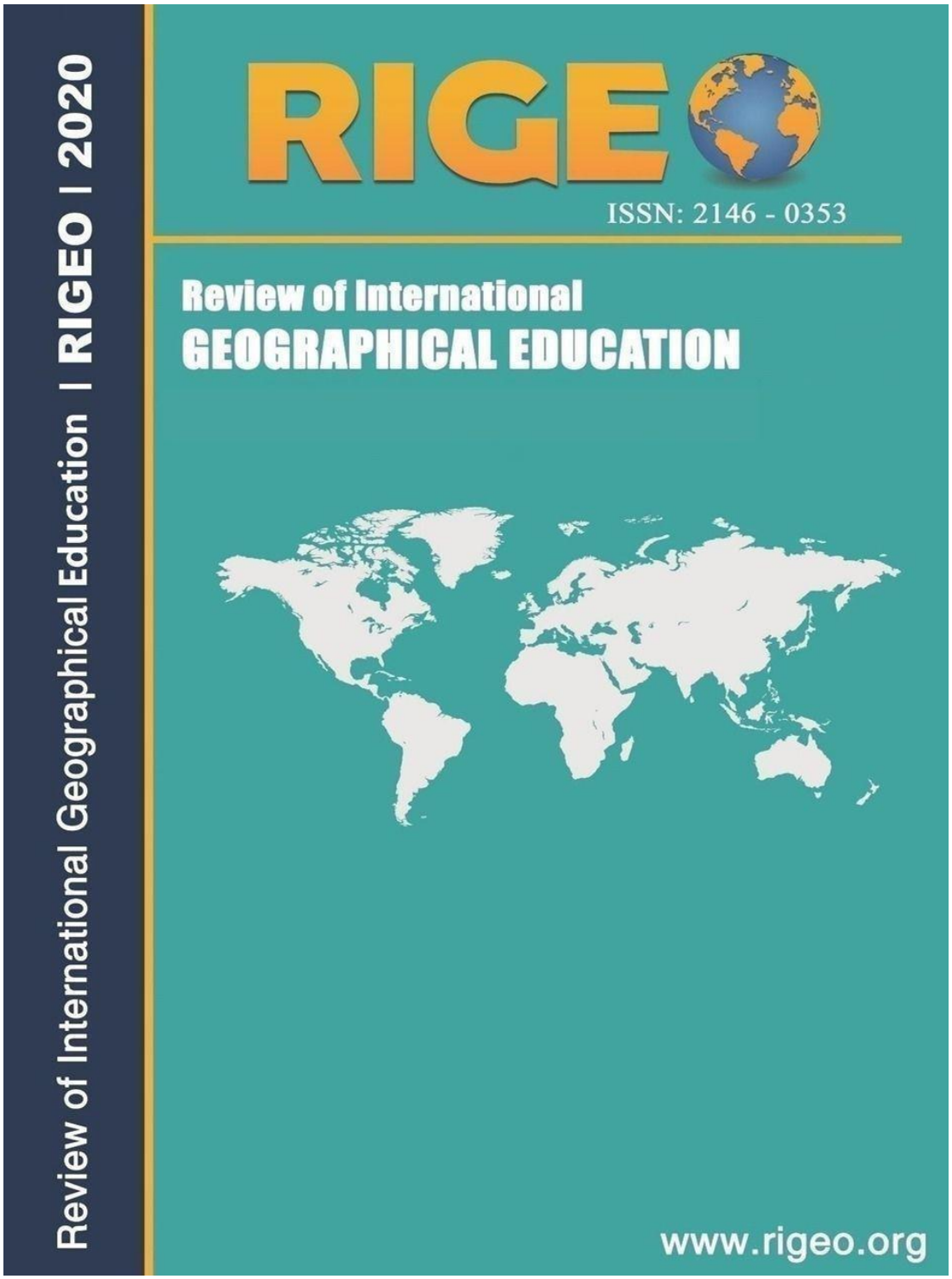
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The Second Kind of Nonlinear Volterra-Hammerstein Integral Equations: A Change to the Successive Approximation Method

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Abstract. In order to solve second-kind nonlinear Volterra-Hammerstein integral equations, this study offers a variant of the successive approximation approach that makes use of the projection operator. The proof of convergence to the precise solution occurs under certain circumstances in the series of iterated solutions in this work. Some numerical examples have shown the applicability of this modification. The method's superiority is further shown by comparisons with other approaches.

AMS Subject Classification: 65D15; 65R20
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1. Introduction

In this paper we consider the following nonlinear Volterra Hammerstein integral equation

$$y(s) = x(s) + \int_0^s k(s, t)g(t, y(t))\, dt, \qquad s \in [0, 1], \tag{1}$$

where x, k and g are known functions. The function $g(t, y(t))$ is nonlinear in the unknown function y.

Applying Green's function method on a nonlinear boundary value problem leads to an integral equation of Hammerstein type. Also in studying some phenomena in various branches of science and engineering, one may encounter integral equations of Hammerstein type [4, 5].

There are numerous papers which are devoted to study the solution of these family of nonlinear integral equations. The projection methods such as Galerkin and collocation methods and their variants are well known and popular methods which have been used to solve the Hammerstein integral equations numerically see for example [16, 2, 12, 21, 6, 13, 24]. These methods convert the integral equation into a system of algebraic equations which is usually solved by an iterative technique. Some modifications of these methods have allocated much attention to it- self, for example in [17], Kumar and Sloan presented a new collocation method to solve Fredhlom-Hammerstein integral equations. Several papers have used the Kumar-Sloan technique by different basis functions such as orthogonal functions and wavelets [9, 10, 23]. In addition to these methods, other approaches have been used to estimate the solution of the Hammerstein integral equations such as degenerate kernel method [14], iterated degenerate kernel method [15], a variation of the Nystrom method [18], Adomian decomposition method [1], etc.

Successive approximation method (Picard iterative method) is a classic approach which can handle both linear and nonlinear problems. One of the drawbacks of this method is it's increasing amount of computations in the first few iterations. This causes the algorithm to be stopped in the beginning of the implementation by a software like Maple. Here we have proposed a modification in successive approximation method to overcome this problem by using a projection operator in the iterative method.

The paper is organized as follows:

In Section 2, preliminary mathematics about successive approximation method and the best approximation solution in $L^2[0, 1]$ is presented. In Section 3, we present the modified successive approximation method. The convergence discussion is given in Section 4. Finally, in Section 5, some numerical examples are presented to confirm effectiveness and applicability of the approach.

2. Mathematical Preliminary

2.1 Successive approximation method

Consider the Volterra-Hammerstein integral equation (1), where k, g, x are continuous functions. One of the ways to obtain some approximations to the exact solution is to use the following recurrence relation

$$\begin{aligned} &y_0(s) := x(s), \\ &y_{i+1}(s) := x(s) + \int_0^s k(s,t)g(t,y_i(t))\,dt, \quad i = 0, 1, \cdots . \end{aligned}$$

Here the function g satisfies the Lipschitz condition with respect to it's second variable

$$|g(s,t) - g(s,u)| \leq \gamma|t-u|, s \in [0,1],$$

where γ is independent of s, t and u. The convergence of the generated sequence $\{y_i\}_{i=0}^{\infty}$ and whereby existence and uniqueness of the exact solution of (1) is guarantied by the Banach fixed point theorem [3].
We define the nonlinear operators, $T : L^2[0,1] \to L^2[0,1]$ and $G : L^2[0,1] \to C[0,1]$ as follows

$$Ty(s) = x(s) + \int_0^s k(s,t)y(t)dt, \quad s \in [0,1], \tag{2}$$

and

$$G(x)(t) = g(t,x(t)), \quad t \in [0,1]. \tag{3}$$

Eq. (1) and the above iteration method can be represented in the operator form, respectively as

$$y(s) = TG(y)(s), \quad s \in [0,1], \tag{4}$$

$$\begin{aligned} &y_0(s) := x(s), \\ &y_{i+1}(s) := TG(y_i)(s), \quad i = 0, 1, \cdots . \end{aligned} \tag{5}$$

2.2 The best approximation in $L^2[0, 1]$

Let the sequence $\{\varphi_i\}_{i=0}^{\infty}$ be a complete orthonormal set of functions in $L^2[0, 1]$ in which φ_i is a polynomial of degree i, $i = 0, 1, \cdots$, and $P_m, m \geq 1$ denotes the space of polynomial functions of degree $\leq m$. Let the inner product and norm of space $L^2[0, 1]$ be respectively

$$(u, v) = \int_0^1 u(t)v(t)dt, \quad \|u\|_{L^2} = \sqrt{(u, u)}, \text{ for all } u, v \in L^2[0\ 1],$$

For any $x \in L^2[0, 1]$, the approximate function $x_m = \sum_{k=1}^{m}(x, \varphi_k)\varphi_k$ is called the best approximation for x in P_m which satisfies the following optimal condition

$$\|x - x_m\|_{L^2} = \inf_{u \in P_m} \|x - u\|_{L^2}. \tag{6}$$

The following theorem shows that by increasing the value of m, a better approximations for x can be obtained.

Theorem 2.2.1. [8] *Let $\{\varphi_i\}_{i=0}^{\infty}$ be a complete orthonormal sequence of functions in $L^2[0, 1]$. For any $x \in L^2[0, 1]$, the sequence $\{x_i\}_{i=0}^{\infty}$ defined by $x_m = \sum_{k=1}^{m}(x, \varphi_k)\varphi_k$ converges uniformly to x.*

The map P_m from $L^2[0, 1]$ into P_m which is defined as $P_m x = \sum_{k=1}^{m}(x, \varphi_k)\varphi_k$ is an orthogonal projection operator. Hence, Theorem 2.2.1, results in

$$\|x - P_m x\|_{L^2} \to 0 \text{ as } m \to \infty, \text{ for all } x \in L^2[0, 1]. \tag{7}$$

The inner product of $L^2[0, 1]$ can be approximated by the following discrete inner product which is obtained by using the Gauss-Lobatto quadrature

$$(u, v)_m = \sum_{j=0}^{m} u(s_j)v(s_j)w_j, \tag{8}$$

where $w_j = \frac{1}{m(m+1)\,[L_m(s_j)]^2}, j = 0, \cdots, m$ and the set $\{s_j\}_{j=0}^{m}$ is the shifted Legendre Gauss-Lobatto nodes in [0, 1].

According to [7], if x belongs to the Sobolev space $H^N(0, 1) = \{u | \frac{d^k}{dt^k} u \in L^2[0, 1], 0 \le k \le N\}$, the following error estimate between the continuous and discrete inner products holds

$$|(x, \varphi) - (x, \varphi)_m| \le c_1 m^{-N} \|x\|_{HN} \|\varphi\|_{L^2}, \forall\ \varphi \in \mathrm{P}_m, \tag{9}$$

where $\|x\|_{HN} = (\sum_{k=0}^{N} \|u^{(k)}\|_{L^2}^2)^{\frac{1}{2}}$, and c_1 is a positive constant independent of N, m,φ, x. Obviously, $\|x\|_{L^2} \le \|x\|_{HN}$ for all $x \in H^N(0, 1)$. We define the operator Q_m from $L^2[0, 1]$ to P_m as

$$Q_m x = \sum_{k=0}^{m} (x, \varphi_k)_m \varphi_k. \tag{10}$$

With the aid of the assumptions of Theorem 2.2.1, and (9), for all $x \in H^N(0, 1)$ we obtain

$$\begin{aligned} \|P_m x - Q_m x\|_{L^2} &= \|\sum_{k=0}^{m} ((x, \varphi_k) - (x, \varphi_k)_m)\varphi_k\|_{L^2} \\ &\le \sum_{k=0}^{m} |(x, \varphi_k) - (x, \varphi_k)_m| \\ &\le c_1(m + 1)m^{-N} \|x\|_{HN}. \end{aligned} \tag{11}$$

Therefore for all $x \in H^N(0, 1)$ and $m \ge 1$,

$$\begin{aligned} \|Q_m x\|_{L^2} &\le \|P_m x - Q_m x\|_{L^2} + \|P_m x\|_{L^2} \\ &\le c_1(m + 1)m^{-N} \|x\|_{HN} + \|P_m\| \|x\|_{L^2} \end{aligned}$$

As $\|P_m\| = 1$, we have

$$\|Q_m x\|_{L^2} \le (c_1(m + 1)m^{-N} + 1)\|x\|_{HN},$$

and consequently

$$\|Q_m\| \le q_m := 1 + c_1(m + 1)m^{-N}, \text{for all } m \ge 1. \tag{12}$$

It is easily seen that according to (11) and (7)

$$\|x - Q_m x\|_{L^2} \to 0 \text{ as } m \to \infty, \text{ for all } x \in H^N(0, 1), N \ge 2. \tag{13}$$

Furthermore, according to (12)

$$q_m \to 1 \text{ as } m \to \infty. \tag{14}$$

In this paper, the complete orthonormal sequence is considered to be $\{\varphi_i\}_{i=0}^{\infty}$, where $\varphi_i(t) := \frac{\tilde{L}_i(t)}{\|\tilde{L}_i\|}$, $t \in [0, 1]$, and $\tilde{L}_i(t) = L_i(2t - 1)$, $i = 0, 1, 2, \ldots$, are shifted Legendre polynomials. The functions $L_i(t)$, $i = 0, 1, 2, \ldots, t \in [-1, 1]$, are the well-known Legendre polynomials whichcan be obtained recursively, as follows

$$L_0(t) := 1, L_1(t) := t,$$
$$L_{j+1}(t) := \frac{2j+1}{j+1} t L_j(t) - \frac{j}{j+1} L_{j-1}(t), j \geq 1.$$

3. Modified Successive Approximation Method

Usually, the number of terms in iterative functions grows rapidly in each stage by increasing the number of iterations. Therefore the algorithm may be failed when it is executed by any software like Maple. Therefore we need to modify the method to prevent the growth of computations. In the following algorithm, we modify the successive approximation method slightly. In each step, after calculating the new iterative function we approximate it by its best approximation in P_m and in this way we control the terms of the iterative functions.

Algorithm 1.
Let the functions k, g, x, number m and the precision ε be given and $s \in [0, 1]$.

step 0) Set $\bar{z}_{0,m}(s) := 0$;
for $n = 1, \cdots$ while $\|\bar{z}_{n,m} - \bar{z}_{n-1,m}\|_{L^2} \geq \varepsilon$ do
step 1) $\hat{z}_{n,m}(s) := x(s) + \int_0^s k(s, t)\bar{z}_{n,m}(t)dt$;
step 2) $r_{n,m}(s) := g(s, \hat{z}_{n,m}(s))$;
step 3) $\bar{z}_{n+1,m}(s) := \sum_{i=0}^{m} (r_{n,m}, \varphi_i)\varphi_i(s)$;
end for,
step 4) Set $\bar{y}_{n,m}(s) := x(s) + \int_0^s k(s, t)\bar{z}_{n+1,m}(t)dt$.

According to the definition of operators T, G and P_m, Algorithm 1 can be represented as follows.
Let the functions k, g, x, number m and the precision ε are given and $s \in [0, 1]$.

Set $\bar{z}_{0,m}(s) := 0$;
for $n = 1, \cdots$ while$\|\bar{z}_{n,m} - \bar{z}_{n-1,m}\|_{L^2} \geq \varepsilon$ do
$\bar{z}_{n+1,m}(s) := P_m GT(\bar{z}_{n,m})(s)$;
Set $\bar{y}_{n,m}(s) := T(\bar{z}_{n+1,m})(s)$.

In fact, if the solution of the following equation exists then it can be approximated using the sequence $\{\bar{z}_{n,m}\}_{n=0}^{\infty}$.

$$\bar{z}_m = P_m GT(\bar{z}_m), \quad m \geq 1, \bar{z}_m \in \mathrm{P}_m. \tag{15}$$

Remark 3.1. *Kumar and Sloan [17] proposed a different collocation method based on a modification of the original problem. They used the collocation method to find the solution of the following modified equation instead of* (4)

$$z = GT(z). \tag{16}$$

To compare with (15)*, the collocation equation related to* (16) *was written as*

$$z_n = P_n GT(z_n),$$

where P_n is an interpolatory operator.

In each iteration of the Algorithm 1, m inner products are performed which causes some computational drawbacks, hence we replace the continuous inner product by the discrete inner product (8) which results in the following algorithm.

Algorithm 2: Modified successive approximation
Let functions k, g, x, number m and the precision ε be given and $s \in [0, 1]$.
Set $z_{0,m}(s) := 0$;
for $n = 1, \cdots$ while$\|z_{n,m} - z_{n-1,m}\|_{L^2} \geq \varepsilon$ do
$z_{n+1,m}(s) := Q_m GT(z_{n,m})(s)$;
Set $y_{n,m}(s) := T(z_{n+1,m})(s)$.
Indeed, under certain conditions, Algorithm 2, finds the fixed point of the following equation

$$z_m = Q_m GT(z_m), \quad m \geq 1 \quad z_m \in \mathrm{P}_m. \tag{17}$$

For later use, we define the sequence $\{y_m\}_{m=1}^{\infty} \subset L^2[0, 1]$ as

$$y_m = T(z_m), \quad m \geq 1. \tag{18}$$

4. Error Analysis

Let the following assumptions be satisfied by functions x, k, g.
$H1$: $x \in H^N(0, 1)$.
$H2$: The kernel function $k(s, t)$ is in $H^N((0, 1)^2)$.
$H3$: The function $g(t, y)$ is in $H^N((0, 1) \times \mathrm{R})$.
$H4$: $\gamma := \sup_{(s,u)\in([0,1]\times \mathrm{R})} \frac{\partial g(s,u)}{\partial u} < \infty$.
A straight result of the assumptions $H3$ and $H4$ is that the function g satisfies a Lipschitz condition in the second variable with the Lipschitz constant γ,

$$|g(s, t) - g(s, u)| \leq \gamma|t - u|, \text{ for all } s \in [0, 1] \text{ and } t, u \in \mathrm{R}.$$

Theorem 4.1. *Let assumptions $H1$-$H4$ be satisfied. Then the operators G, GT, P_mGT and Q_mGT, $m \geq 1$ satisfy a Lipschitz condition.*

Proof. Let y_1 and y_2 be two arbitrary functions in H^N $(0, 1)$. Since the function g satisfies the Lipschitz condition, we have

$$|G(y_1)(t)-G(y_2)(t)| = |g(t, y_1(t))-g(t, y_2(t))| \leq \gamma|y_1(t)-y_2(t)|, \text{ for all } t \in [0, 1].$$

This immediately implies that

$$\|G(y_1) - G(y_2)\|_{L^2} \leq \gamma\|y_1 - y_2\|_{L^2}. \tag{19}$$

To prove the Lipschitz condition for GT, from Schwartz inequality for $t \in [0, 1]$ we have

$$\begin{aligned} |Ty_1(t) - Ty_2(t)|^2 &\leq \left[\int_0^t |k(t, s)(y_1(s) - y_2(s))|ds\right]^2 \\ &\leq \int_0^t |k(t, s)|^2 ds \int_0^t |y_1(s) - y_2(s)|^2 ds \\ &\leq \int_0^t |k(t, s)|^2 ds \int_0^1 |y_1(s) - y_2(s)|^2 ds. \end{aligned} \tag{20}$$

Taking the integral of both side of (20) over interval [0, 1], and using Schwartz inequality we obtain

$$\|Ty_1 - Ty_2\|_{L^2} \leq K\|y_1 - y_2\|_{L^2}, \tag{21}$$

where $K^2 = \int_0^1 \int_0^t |k(t,s)|^2 dsdt$. Using (19) and (21), the Lipschitz continuity of the operator GT is inferred,

$$\|GT(y_1) - GT(y_2)\|_{L^2} \leq \gamma\|Ty_1 - Ty_2\|_{L^2} \leq K\gamma\|y_1 - y_2\|_{L^2}. \tag{22}$$

The linear operator P_m, $m \geq 1$ is an orthogonal projection, hence $\|P_m\| = 1$ and according to the relation (22), we conclude that

$$\|P_mGT(y_1) - P_mGT(y_2)\|_{L^2} \leq \|GT(y_1) - GT(y_2)\|_{L^2} \leq K\gamma\|y_1 - y_2\|_{L^2}. \tag{23}$$

Finally, the Lipschitz continuity for Q_mGT, $m \geq 1$ follows from (22) and (12),

$$\|Q_mGT(y_1) - Q_mGT(y_2)\|_{L^2} \leq q_m\|GT(y_1) - GT(y_2)\|_{L^2} \leq q_mK\gamma\|y_1 - y_2\|_{L^2}. \tag{24}$$

The following theorems establish the convergence of the sequence generated by the modified successive approximation method and (17). Q

Theorem 4.2. *Let assumptions H1-H4 be satisfied and* $0 \leq K\gamma < 1$, *then the sequence* $z_{n,m}$ *generated by the modified successive approxi- mation method is convergent to the solution of equation* (17) *and con- sequently the sequence* $\{y_{n,m}\}$ *defined in the algorithm is convergent to* y_m *and satisfies the following inequality*

$$\|y_{n,m} - y_m\|_{L^2} \leq K\frac{(K\gamma)^n}{1 - K\gamma}\|G(x)\|_{L^2}. \tag{25}$$

Proof. Since $0 \leq K\gamma < 1$, according to (12) there is an $M > 0$ sufficiently large so that $0 \leq q_mK\gamma < 1$, $m \geq M$, thus the operator

$Q_m GT$ is a contractive map on $H^N(0, 1)$ for $m \geq M$. Then by Banach fixed point theorem [3] it has a unique fixed point z_m on $H^N[0, 1]$ and the sequence $\{z_{n,m}\}$ generated by the algorithm is convergent to it.
The convergence of $\{y_{n,m}\}$ to y_m as $n \to \infty$ is obtained by using the following inequality

$$\|y_{n,m} - y_m\|_{L^2} = \|Tz_{n,m} - Tz_m\|_{L^2} \leq K\|z_{n,m} - z_m\|_{L^2}. \tag{26}$$

To prove the inequality (25), according to the Banach fixed point theorem [3] the sequence $\{z_{n,m}\}$ satisfies

$$\|z_{n,m} - z_m\|_{L^2} \leq \frac{(K\gamma)^n}{1 - K\gamma}\|z_{0,m} - z_{1,m}\|_{L^2}.$$

Since $z_{0,m}(s) = 0$ and $T(z_{0,m})(s) = x(s)$, we have

$$\begin{aligned}\|z_{n,m} - z_m\| &\leq \tfrac{(K\gamma)^n}{1-K\gamma}\|z_{1,m}\|_{L^2} \\ &\leq \tfrac{(K\gamma)^n}{1-K\gamma}\|Q_m GT(z_{0,m})\|_{L^2} \\ &\leq \tfrac{(K\gamma)^n}{1-K\gamma}\|GT(z_{0,m})\|_{L^2} \\ &= \tfrac{(K\gamma)^n}{1-K\gamma}\|G(x)\|_{L^2}.\end{aligned}$$

Now by using (26), the inequality is obtained

$$\|y_{n,m} - y_m\|_{L^2} \leq K\|z_{n,m} - z_m\|_{L^2} \leq K\frac{(K\gamma)^n}{1 - K\gamma}\|G(x)\|_{L^2}. \qquad \square$$

Theorem 4.3. *Let assumptions H1-H4 be satisfied and* $0 \leq K\gamma < 1$, *then* $\{y_m\}_{m=1}^{\infty}$ *defined by* (18) *is convergent to* y^*, *which is the exact solution of* (1).

Proof. By defining $z^* := Gy^*$, we get $Tz^* = TGy^* = y^*$, these lead to $z^* = GTz^*$. By virtue of (21) and (24), we have

$$\|y_m - y^*\|_{L^2} = \|Tz_m - TGy^*\|_{L^2} = \|Tz_m - Tz^*\|_{L^2} \leq K\|z_m - z^*\|_{L^2}, \tag{27}$$

and for all $m \geq M$

$$\begin{aligned}\|z_m - z^*\|_{L^2} &= \|Q_m GT(z_m) - z^*\|_{L^2} \\ &= \|Q_m GT(z_m) - Q_m GT(z^*) + Q_m GT(z^*) - z^*\|_{L^2} \\ &\leq \|Q_m GT(z_m) - Q_m GT(z^*)\|_{L^2} + \|Q_m(z^*) - z^*\|_{L^2} \\ &\leq q_m K\gamma \|z_m - z^*\|_{L^2} + \|Q_m(z^*) - z^*\|_{L^2}.\end{aligned} \tag{28}$$

By using (27) and (28) we have

$$\|y_m - y^*\|_{L^2} \leq K\|z_m - z^*\|_{L^2} \leq \frac{K}{1 - q_m K\gamma}\|Q_m z^* - z^*\|_{L^2},$$

and the convergence is derived immediately from (13) and (14). Q

5. Numerical Examples

In all of the following examples, for various amounts of m, we compute the absolute errors in the following specified norms

$$E_2 = \left(\sum_{j=1}^{N} (y_{exact}(x_j) - y_{app}(x_j))^2\right)^{\frac{1}{2}},$$

$$E_\infty = \max_{j=1..N} \left(|y_{exact}(x_j) - y_{app}(x_j)|\right).$$

In Example 1, the convergence conditions of Theorem 4.2, are satisfied. However, in Example 2 and 3 although some of the conditions of the Theorem 4.2, are not established, but the results show the convergence of the method. This confirms that the conditions of Theorem 4.2, are sufficient and not necessary.
Computations are carried out in Maple V. 15 software, with hardware configuration 32 bit intel Core 2 Duo CPU and 2 GB of RAM.

Example 1. Consider the following nonlinear Volterra-Hammerstein integral equation of the second kind

$$y(s) = x(s) - \int_0^s s^3 \cos(t)\cos(y(t))dt, \qquad 0 \leq s \leq 1, \tag{29}$$

where $x(s) = s - 6\cos(s) - s^3\cos(s)\sin(s) - 3s^2\cos^2(s) + 6\cos^2(s) + 6s\cos(s)\sin(s)$. The exact solution is $y(s) = s$.

Here $K := \left(\int_0^1 \int_0^1 |s\cos(t)|^2 \, dsdt\right)^{\frac{1}{2}} = 0.08434$ and $\gamma := 1$, so $K\gamma = 0.08434 < 1$ and according to the Theorem 4.2, the method is convergent. The results of the method are presented in Table 1.

The results emphasize that when we increase the value of m the approximate solution can quickly converge to the exact solution. The absolute error of approximate solution for $m = 15$ is depicted in Figure 1.

Example 2. Consider the nonlinear Volterra-Hammerstein integral equation of the second kind as follows:

$$y(s) = 1 + \sin^2(s) - 3\int_0^s \sin(s - t)y(t)^2 \, dt, \qquad 0 \le s \le 1, \qquad (30)$$

which has the exact solution $y(s) = \cos(s)$.

The method has been applied for various values of m. The results are given in Table 2.

The table shows the rapid decrease in the error by increasing m. To affirm the precision of our method, in Figure 2, the absolute error ofthe approximate solution corresponding to $m = 15$ is plotted. In Table3 we have represented the error of approximations for $m = 20$, which are obtained by our approach and Radial basis function method (RBF) [22] and Single-term Walsh series method (STWS) [23].

Example 3. Consider the following nonlinear Volterra-Hammerstein integral equation of the second kind given in [20]:

$$y(s) = \frac{3}{2} - \frac{1}{2}e^{-2s} - \int_0^s y(t) + y^2(t) \, dt, \qquad 0 \le s \le 1, \qquad (31)$$

which has the exact solution $y(s) = e^{-s}$.

The method has been implemented and the results have been summarized in Table 4. Looking at the table, it can be seen that the absolute error is rapidly reduced by increasing m. For comparison with the other methods in Table 5, the absolute errors of present method and the other methods are given. It shows the effectiveness and accuracy of our method in comparison with the other methods. The absolute error of approximate solution for $m = 15$ is depicted in Figure 3.

Table 1: The E_2 and E_∞ errors with various m in Example 1.

m	number of iterations	E_∞	E_2
5	5	2.23108 E -7	8.71336 E -8
10	10	5.87671 E -15	2.32414 E -15
15	16	5.77612 E -24	2.26498 E -24
20	20	4.12562 E -33	1.61230 E -33

Table 2: The E_2 and E_∞ errors and CPU time with various m, in Example 2.

m	Number of iterations	E_∞	E_2	CPU time
5	8	1.54592 E -7	8.28632 E -8	2.6364
10	10	4.05377 E -14	2.26688 E -14	4.5708
15	14	7.60379 E -22	4.12729 E -22	13.2600
20	36	8.89121 E -30	4.48100 E -30	67.2364

Table 3: Comparison between absolute error of the present method, RBF and STWS methods in example 2.

t	Our method $m = 20$	RBF method $N = 20$	STWS method $m = 80$
0.2	5.93151 E-30	2.2529 E -16	1.0 E -5
0.4	7.98636 E-30	1.1229 E -16	1.0 E -5
0.6	7.93769 E-30	7.5228 E -17	2.0 E -5
0.8	5.97590 E-30	6.8774 E -17	2.0 E -5
1.0	7.39411 E-46	2.0931 E -16	2.0 E -5

Table 4: The E_2 and E_∞ errors and CPU time with various m in Example 3.

m	Number of iterations	E_∞	E_2	CPU time
5	15	9.22560 E -7	5.79565 E -7	1.6380
10	21	2.82371 E -13	1.76812 E -13	3.3384
15	29	1.16352 E -20	7.29077 E -21	7.9717
20	35	1.14054 E -28	7.13981 E -29	15.1945

Table 5: Comparing our approach and the methods in [18, 19] for Example 3.

m	Our method	Bernstein operational matrices method [19]	TF method[18]
4	8.49274 E -6	6.819 E-5	3.738 E-3
8	9.48008 E -11	8.429 E -8	9.3701 E -4
16	2.02022 E -22	* < E -13	2.344 E -4
32	4.60349 E -48	* < E -13	2.374 E -3

*Reported up to 13 digits

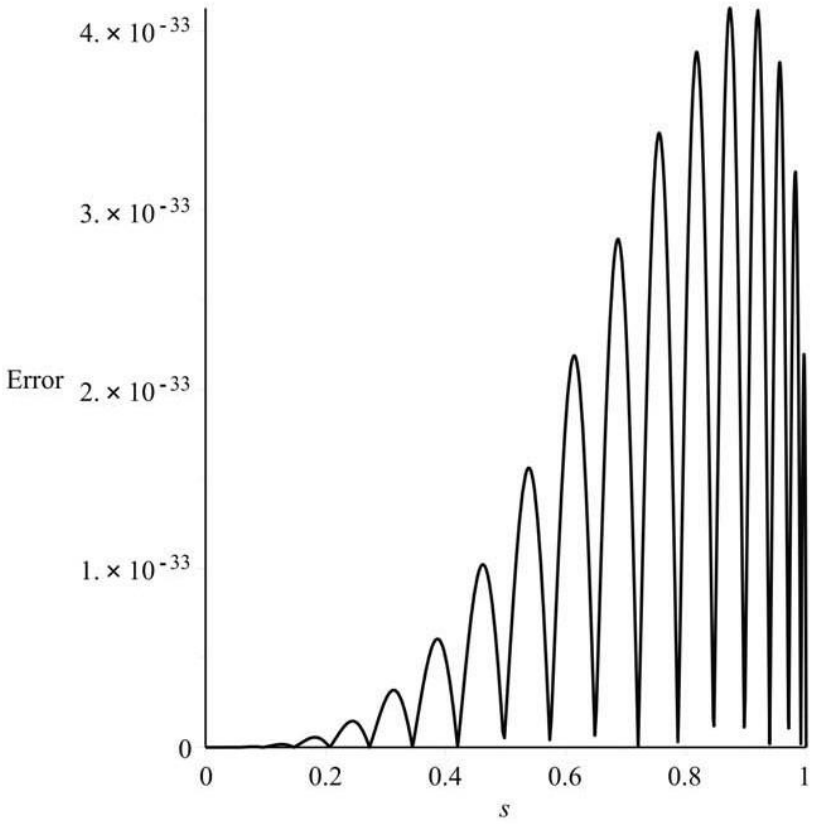
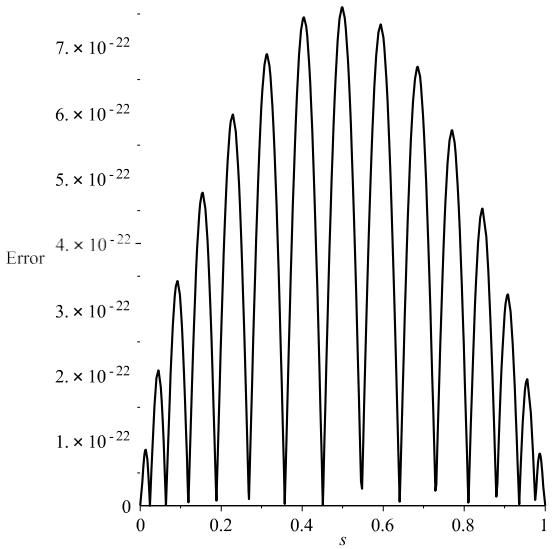
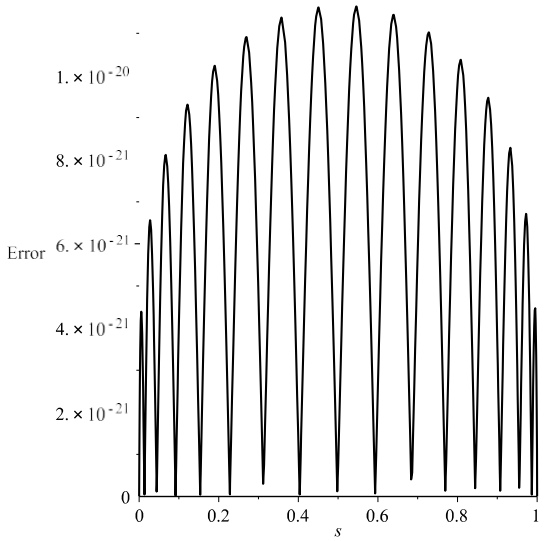


Figure 1: The absolute error between approximate solution and the exact solution with $m = 15$ in Example 1.



+iJXUH 2! 9KH DEVROXWH HUURU EHWZHHQ DSSUR[iPDWH VROXWiRQ DQG WKH H[DFW VROXWiRQ ZiWK > # 15 iQ *[DPSOH 2.



+iJXUH 3! 9KH DEVROXWH HUURU EHWZHHQ DSSUR[iPDWH VROXWiRQ DQG WKH H[DFW VROXWiRQ ZiWK > # 15 iQ *[DPSOH 3.

6. Conclusion

In this study, we have proposed a modification to successive approximation method by using the projection of the iteration functions in each stage. The applicability of the method is shown with the implementation of the method in some examples. By comparison of the numerical results of the present method with the exact solution and some other methods, the performance and superiority of the method have been confirmed. The convergence analysis of the method has also been discussed.

References

[1] G. Adomian, Analytic solution of nonlinear integral equations of hammerstein type, *Appl. Math. Lett.,* 11 (1998), 127-130.

[2] K. Atkinson and J. Flores, A discrete collocation method for nonlinear integral equations, *IMA j. Num. Anal.*, 13 (1993), 195-213.

[3] K. Atkinson and W. Han, *Theoretical Numerical Analysis, a Func- tional Analysis Frameworks*, Third edition, Texts in applied mathematics, Springer, (2009).

[4] S. M. Berman and A. L. Stewart, A nonlinear integral equation for visual impedance, *Biol. Cybernetics.* 33 (1979), 137-141.

[5] A. M. Bica, M. Curila, and S. Curila, About a numerical method of successive interpolations for functional Hammerstein integral equations, *J. Comput. Appl. Math.,* 236 (2012), 2005-2024.

[6] H. Brunner, On implicitly linear and iterated collocation methods for Hammerstein integral equations, *J. Integral. Equa. Appl.,* 3 (1991), 475-488.

[7] C. Canuto, M. Y. Hussaini, A. Quarteroni, and T. A. Zang, *Spectral Methods: Fundamentals in Single Domains*, Springer, Berlin, 2006.

[8] L. M. Delves and J. L. Mohamed,*Computational methods for integral equations*, Cambridge university press, (1988).

[9] G. N. Elnagar and M. Razzaghi, A pseudospectral method for Hammerstein equations, *J. Math. Anal. Appl.,* 199 (1996), 579-591.

[10] G. N. Elnagar and M. Kazemi, Chebyshev spectral solution of nonlinear Volterra-Hammerstein integral equations, *J. Comput. Appl. Math.,* 76 (1996), 147-158.

[11] F. Ghoreishi and M. Hadizadeh, Numerical computation of the Tau approximation for the Volterra-Hammerstein integral equations, *Numer. Algor.,* 52 (2009), 541-559.

[12] I. G. Graham, S. Joe, and I. H. Sloan, Iterated Galerkin versus iterated collocation for integral equations of the second kind, *IMA j. Num. Anal.,* 5 (1985), 355-369.

[13] C. H. Hsiao, Hybrid function method for solving Fredholm and Volterra integral equations of the second kind, *J. Comput. Appl. Math.,* 230 (2009), 59-68.

[14] H. Kaneko and Y. Xu, Degenerate kernel method for Hammerstein equations, *Math. Comput.,* 56 (1991), 141-148.

[15] H. Kaneko, P. Padilla, and Y. Xu, Superconvergence of the iterated degenerate kernel method, *Appl. Anal.,* 80 (2001), 331-351.

[16] A. Karoui and A. Jawahdou, Existence and approximate L^p and continuous solutions of nonlinear integral equations of the Hammerstein and Volterra types, *Appl. Math. Comput.,* 216 (2010), 2077-2091.

[17] S. Kumar and I. H. Sloan, A new collocation-type method for Hammerstein integral equations, *Math. Comp.,* 48 (1987), 123-199.

[18] L. J. Lardy, A variation of Nystroms method for Hammerstein integral equations, *J. Integ. Equ.,* 3 (1982), 123-129.

[19] K. Maleknejad, H. Almasieh, and M. Roodaki, Triangular functions (TF) method for the solution of nonlinear Volterra–Fredholm integral equations. *Commun. Nonlinear Sci. Numer. Simula.,* 15 (2010), 3293-3298.

[20] K. Maleknejad, E. Hashemizadeh, and B. Basirat, Computational method based on Bernstein operational matrices for nonlinear Volterra–Fredholm –Hammerstein integral equations, *Commun. Nonlinear Sci. Numer. Simula.,* 17 (2012), 52-61.

[21] Y. Ordokhani and M. Razzaghi, Solution of nonlinear Volterra–Fredholm –Hammerstein integral equations via a collocation method and rationalized Haar functions, *Appl. Math. Lett.,* 21 (2008), 4-9.

[22] K. Parand and J. A. Rad, Numerical solution of nonlinear Volterra Fred Holm Hammerstein integral equations via collocation method based on radial basis functions, *Appl. Math. Comput.,* 218 (2012), 5292-5309.

[23] B. Sepehrian and M. Razzaghi, Solution of nonlinear Volterra-Hammer-stein integral equations via single-term walsh series method, *Mathl. Probl. Eng.,* 5 (2005), 547-554.

[24] I. H. Sloan, The galerkin method for integral equations of the second kind, *IMA J. Num. Anal.,* 4 (1984), 9-17.