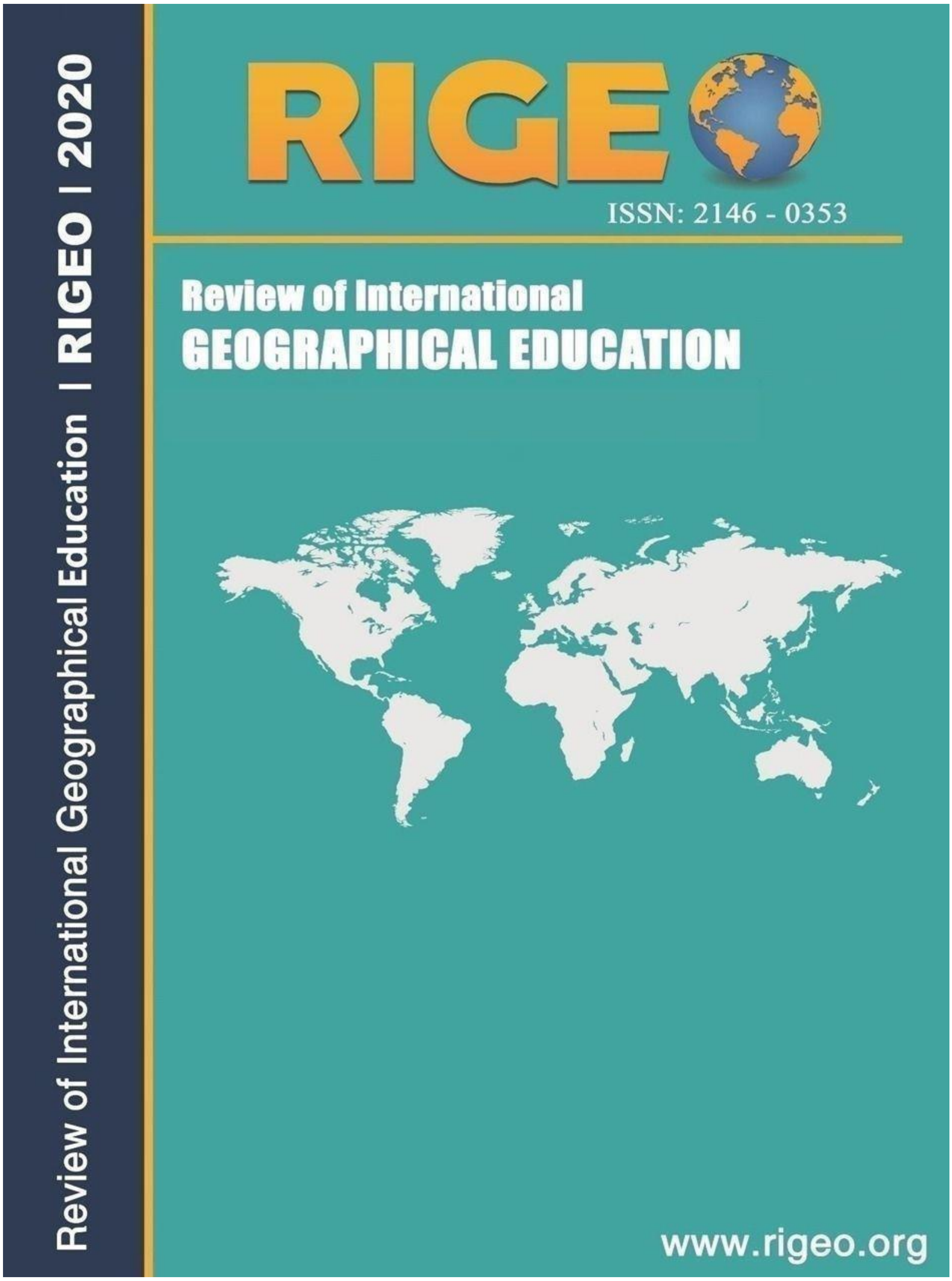
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STASTICAL INFERENCE AND MODELING WITH THE INVERTED LENGTH-BIASED EXPONENTIAL MODEL

K R Girish[1], Dr N Prabhudeva[2], A M Girija[3]

Asst. Professor, Professor & HOD, Asst. Professor

girishhspt@gmail.com[1] , drnprabhudeva@gmail.com[2], girijaam09@gmail.com[3]

Department of Mathematics, Proudhadevaraya Institute of Technology, Abheraj Baldota Rd, Indiranagar, Hosapete, Karnataka-583225

1. Introduction

Length-biased exponential (LBE) or moment exponential (ME) distribution is considered as one of the most important univariate and parametric models. It is commonly utilized in the analysis of data collected throughout a lifespan and in problems connected to the modeling of failure processes. There is much to be said for a flexible lifespan distribution model, and this one may be a suitable fit for some sets of failure data. Reference [1] proposed the LBE with the following PDF and distribution function (CDF):

$$g(x; \alpha)\ \ \alpha^2 x e^{-\alpha x}; \qquad x \geq 0, \alpha > 0, \qquad (1)$$

$$G(x; \alpha)\ \ 1 - (1 + \alpha x)e^{-\alpha x}; \qquad x \geq 0, \alpha > 0, \qquad (2)$$

where α is the scale parameter. Different values of the shape parameter lead to different shapes of density function.

Many authors extended new models from the LBE distribution such as exponentiated ME [2], generalized exponentiated ME [3], and Marshall–Olkin (MO) LBE (MOLBE) distributions [4]. MO Kumaraswamy ME model was discussed in [5].

Several univariate continuous distributions have been extensively used in environmental, engineering, financial, and biomedical sciences, among other areas for modeling lifetime data. However, there is still a strong need for a significant improvement of the classical through different techniques for modeling several data lifetime. In this regard, the inverted (or inverse) (I) distribution is one procedure that explores extra properties of the phenomenon which cannot be produced from noninverted distributions. Applications of inverted distributions include econometrics as well as the engineering sciences as well as biology and survey sampling as well as medical research among others. In the literature, several studies related to inverted distributions have been handled by several researchers; for instance, Reference [6] introduced the I Weibull distribution. Reference [7] studied the I Pareto type 1 distribution. Reference [8] investigated the I Pareto type 2 distribution. Reference [9] handled exponentiated I Weibull distribution. Reference [10] provided the I Lindley distribution. Reference [11] suggested the I Kumaraswamy model. Reference [12] presented the I Nadarajah-Haghighi model.

Reference [13] studied the I power Lomax model. Reference [14] suggested I exponentiated Lomax model. Reference [15] discussed the Weibull I Lomax model. Reference [16] suggested the power transmuted I Rayleigh model. Reference [17] investigated the I Topp–Leone distribution, and half logistic I Topp–Leone distribution was studied in [18].

Our motivation here is (i) introducing a new distribution, referred to as the inverted length-biased exponential (ILBE), (ii) studying some of the main properties, (iii) providing point and interval estimators for the model parameter from complete and censored samples, and (iv) examining its applicability using three real datasets.

The inverted LBE (ILBE) distribution is constructed by using the random variable T $1/X$ where X follows (2). The

ILBE distribution's CDF is described as α

$$F(t;\alpha) = \left(1 + \frac{\alpha}{t}\right) e^{-\alpha/t}; \qquad t \geq 0,\ \alpha > 0. \tag{3}$$

The ILBE distribution's PDF is specified as

$$f(t;\alpha) = \frac{\alpha^2}{t^3} e^{-\alpha/t}; \qquad t \geq 0,\ \alpha > 0. \tag{4}$$

The survival function (SRF) and HZRF of the ILBE distribution are provided by

$$\bar{F}(t;\alpha) = 1 - \left(1 + \frac{\alpha}{t}\right) e^{-\alpha/t},$$

$$h(t;\alpha) = \frac{\alpha^2 e^{-\alpha/t}}{t^3 \left[1 - (1 + \alpha/t) e^{-\alpha/t}\right]} \tag{5}$$

Figure 1 depicts PDF and HZRF plots for the ILBE distribution. According to Figure 1, the density of the suggested distribution is highly flexible in nature and can take on a number of forms, including positively skewed and unimodal. Through the parameter space, the HZRF can take on many forms, such as decreasing, rising, or upside down.

This paper is organized as follows. In Section 2, the basic characteristics of the ILBE distribution are obtained. The MLL estimators for the ILBE model are described in Section 3 and are established on complete and censored samples, accompanied by a simulation analysis. The application to actual data collection is covered in Section 4. Section 5 concludes the paper with some remarks.

2. Fundamental Mathematical Properties of ILBE Distribution

Here, we give some essential properties of the ILBE distribution, like QuF, MOs, PRWMOs, incomplete MOs, and inverse MOs.

2.1. Quantile Function. A generated random number from the ILBE distribution is obtained by solving the following equation numerically:

$$Q(u) = \frac{\alpha}{-1 - W_{-1}\left(-e_{-1}u\right)}, \qquad 0 < u < 1, \qquad (6)$$

where W_{-1} denotes the negative branch of the Lambert W function (i.e., the solution of the equation $W(Z)e^{W(Z)} = z$. The median, say Q_2, is achieved by adjusting $u = 0.5$ in (6), and the first quartile and third quartile, denoted by Q_1 and Q_3, are obtained by setting $u = 0.25$ and 0.75, respectively, in (6). Note that equation (6) is solved numerically by using Mathematica 9.

$$Q_1 = \frac{\alpha}{-1 - W_{-1}\left(-0.25e^{-1}\right)},$$

$$Q_2 = \frac{\alpha}{-1 - W_{-1}\left(-0.5e^{-1}\right)}, \qquad (7)$$

$$Q_3 = \frac{\alpha}{-1 - W_{-1}\left(-0.75e^{-1}\right)}.$$

2.2. Moments. Due to its relevance in any statistical study, we shall give the n-th MO of the ILBE distribution here. For the ILBE model, the n-th MO of T about the origin is computed as follows:

μ_n' E T ^{n} ∞ t^n α $_{32}e^{-\alpha/t}$dt $\alpha_n\Gamma(2-n)$, $n<2$. (8)

0 t

The following formula may be used to determine the MOGF of the ILBE distribution:

$M_x(t)$ μ_n' $\alpha^n\Gamma(2-n)$, $n<2$. $n!$ $n!$ $_{n0}$ (9)

$n0$

The incomplete (IN) MO, say $\varsigma_n(x)$, is

$\varsigma_n(t)$ $\alpha^2 t$ $^0 t^{n-3}e^{-\alpha/t}$dt $\alpha^n c$ $2-n$, α, $n<2$, (10)

t

where, $c(., t)$ is the upper IN gamma function.

Further, the conditional MO, say $\varpi_s(x)$, is

$\varpi_s(t)$ α t e dt α Γ $2-$ n, , $n<2$, 11 t t where $\Gamma(., t)$ is the lower IN gamma function.

For the ILBE distribution, the n-th inverse MO is calculated on the basis:

∞

$\tau_k(x)$ α^2 $t^{-n-3}e^{-\alpha/t}$dt $\alpha^{-k}\Gamma(n+2)$. (12)

0

For n 1, we get the harmonic mean of the ILBE distribution.

The Lorenz and Bonferroni curves are obtained as follows 0.0 0.5 1.0 1.5 2.0 2.5 3.0 0 1 2 3 4 t t

α = 0.9α = 1.8α = = 1.5α = 2.5 0.9α = 1.8 α = 1.2α = 2.0α = 1.2α = 2.0 α = 1.5α = 2.5α

Figure 1: Plots of PDF and HZRF for the ILBE distribution.

$L_F(t)$ ^{1} c 1, α , $\varsigma(t)$ (13)

$E(T)$ t

$L_F(t)$ $c(1, (\alpha/t))$

$B_F(t)$ $F(t)$ $(1+(\alpha/t))e_{-\alpha/t}$. (14)

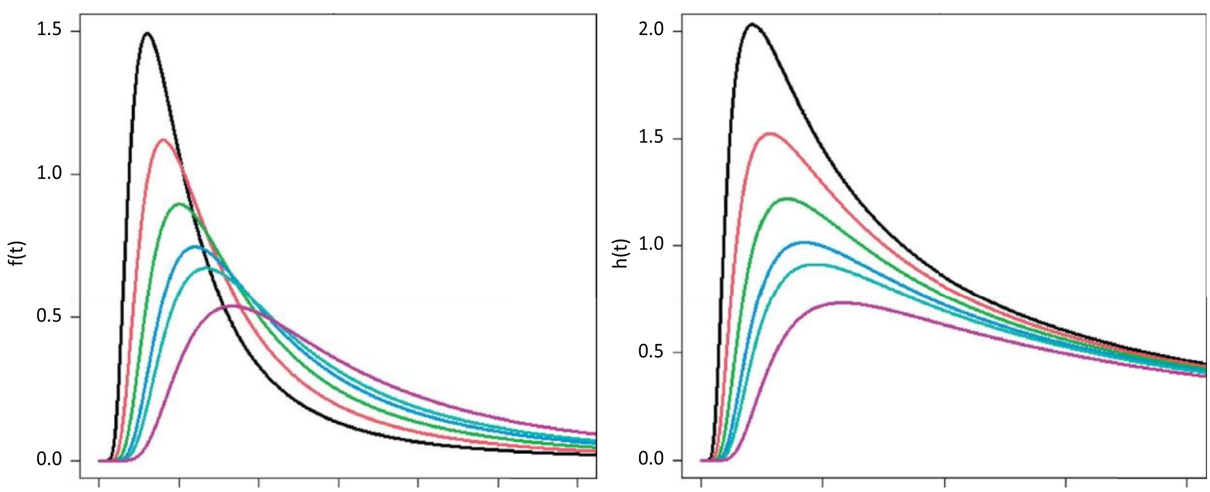
2.3. Order Statistics. Let $T_1, T_2, \ldots, T_n$ be r samples from the ILBE model with order statistics $T_{(1)}, T_{(2)}, \ldots, T_{(n)}$. The PDF of $T_{(k)}$ of order statistics is given by

$n!$ $k-1$ $n-k$ $f_{T(k)}(t)$ $F(t)f(t)(1-F(t))$.

$(k-1)!(n-k)!$

(15)

The PDF of $T_{(k)}$ can be expressed as



$$f_{T(k)}(t) = \frac{n!\alpha^2}{(k-1)!(n-k)!} t^{-3}\left(1+\frac{\alpha}{t}\right)^{k-1} e^{-k\alpha/t}\left[1-\left(1+\frac{\alpha}{t}\right)e^{-\alpha/t}\right]^{n-k}. \quad (16)$$

Particularly, PDF of the first and largest order statistics can be calculated as

$$f_{T(1)}(t) = n\alpha^2 t^{-3} e^{-\alpha/t}\left[1-\left(1+\frac{\alpha}{t}\right)e^{-\alpha/t}\right]^{n-1}, \quad (17)$$

$$f_{T(n)}(t) = n\alpha^2 t^{-3}\left(1+\frac{\alpha}{t}\right)^{n-1} e^{-n\alpha/t}, \quad (18)$$

respectively.

2.4. Mean Residual Life Function. It has an important application of the MOs of residual lifetime function. The MRLS of ILBE distribution is

$$\upsilon(t) = E(T - t \mid T > t) = \frac{1}{F(t)} \int_t^{\infty} x f(x) dt - t \qquad (19)$$

$$= \alpha\Gamma\left(1, \frac{\alpha}{t}\right)\left[1 - \left(1 + \frac{\alpha}{t}\right) e^{-\alpha/t}\right]^{-1} - t.$$

The MINTrepresents the amount of time that has passed after an item has failed, assuming that this failure has occurred. The MINT of ILBE distribution is

$$\varpi(t) = E(T - t \mid T \le t) = t - \frac{1}{F(t)} \int_0^{t} x f(x) dx \qquad (20)$$

$$= t - \alpha c\left(1, \frac{\alpha}{t}\right)\left(1 + \frac{\alpha}{t}\right)^{-1} e^{\alpha/t}.$$

2.5.ProbabilityWeightedMoments. The PRWMOs are often used to investigate additional aspects of the probability distribution. The PRWMOs of the random variable T, denoted by $S_{r,p}$, are defined as

$$S_{r,p} = \int_{-\infty}^{\infty} t^r f(t)[F(t)]^p dt, \qquad (21)$$

where r and p are positive integers. Substituting (3) and (4) into (21) yields the PRWMOs of the ILBE distribution as follows:

Table 1: MLE, ℘, I, [, H, and ב of the ILBE distribution for α 1.2 under TIIC.

n	t_r (%)	MLE	℘	I	90%			95%		
					[	H	ב	[	H	ב
100	60	1.6091	0.4091	0.2111	1.3068	1.9113	0.6045	1.2489	1.9692	0.7203
	80	1.3548	0.1548	0.0546	1.0803	1.6294	0.5490	1.0278	1.6819	0.6542
	100	1.2316	0.0316	0.0263	0.9700	1.4931	0.5231	0.9199	1.5432	0.6233
200	60	1.5648	0.3648	0.1575	1.3372	1.7924	0.4552	1.2937	1.8360	0.5424
	80	1.3195	0.1195	0.0319	1.1125	1.5266	0.4142	1.0728	1.5663	0.4935
	100	1.1993	0.0007	0.0146	1.0020	1.3965	0.3946	0.9642	1.4343	0.4701

	60	1.5777	0.3777	0.1551	1.4154	1.7399	0.3245	1.3844	1.7710	0.3866
300	80	1.3307	0.1307	0.0256	1.1831	1.4784	0.2953	1.1548	1.5067	0.3519
	100	1.2097	0.0097	0.0071	1.0690	1.3504	0.2814	1.0420	1.3773	0.3353

Table 2: MLE, ℘, I, [, H, and ב of the ILBE distribution for α 1.5 under TIIC.

n	t_r (%)	MLE	℘	I	90%			95%		
					[	H	ב	[	H	ב
	60	1.9815	0.4815	0.2846	1.6091	2.3538	0.7447	1.5378	2.4251	0.8873
100	80	1.6720	0.1720	0.0666	1.3332	2.0107	0.6775	1.2683	2.0756	0.8073
	100	1.5196	0.0196	0.0313	1.1969	1.8423	0.6454	1.1351	1.9041	0.7690
	60	1.9785	0.4785	0.2638	1.6907	2.2664	0.5757	1.6356	2.3215	0.6859
200	80	1.6695	0.1695	0.0544	1.4075	1.9315	0.5240	1.3573	1.9817	0.6243
	100	1.5170	0.0170	0.0215	1.2674	1.7665	0.4991	1.2196	1.8143	0.5946
	60	1.9562	0.4562	0.2273	1.7550	2.1573	0.4023	1.7165	2.1958	0.4793
300	80	1.6486	0.1486	0.0359	1.4656	1.8315	0.3659	1.4306	1.8666	0.4359
	100	1.4988	0.0012	0.0114	1.3245	1.6731	0.3487	1.2911	1.7065	0.4154

$$S_{r,p} = \alpha^2 \int_0^\infty t^{r-3}\left(1+\frac{\alpha}{t}\right)^p e^{-(p+1)\alpha/t}\,dt = \sum_{j=0}^{p}\binom{p}{j}\alpha^{j+2}\int_0^\infty t^{r-3-j}e^{-(p+1)\alpha/t}\,dt. \quad (22)$$

As a result of the simplification, the PRWMOs of the ILBE distribution assume the following structure:

$$S_{r,p} = \sum_{j=0}^{p}\binom{p}{j}\frac{\alpha^{r}\,\Gamma(j-r+2)}{(p+1)^{j-r+2}}. \quad (23)$$

3. Statistical Inference

3.1. MLL Estimator Based on TIIC. Assume $T_{(1)}, T_{(2)}, \ldots, T_{(n)}$ are the recorded TIICS of size r, whose lifetimes have the ILBE distribution with PDF (4), and the experiment is completed once the r-th object fails for just some fixed values of r. The log-likelihood function (LLF), according to TIIC, is provided by

$$\ln l_2 \ \ln C + 2r \ln \alpha - 3 \sum_{i1}^{r} \ln t_i - \sum_{i1}^{r} \frac{\alpha}{t_i} + (n-r)\ln\left[1 - \left(1 + \frac{\alpha}{t_r}\right)e^{-\alpha/t_r}\right], \quad (24)$$

and for the sake of simplification, we abbreviate t_i rather than $t_{(i)}$. As a result, the partial derivatives of the LLF with regard to the component of the score $U(\alpha)\ z \ln l_2/z\alpha$ may be computed as follows:

$$U(\alpha) \ \frac{2r}{\alpha} - \sum_{i1}^{r} \frac{1}{t_i} + \frac{(n-r)\alpha e^{-\alpha/t_r}}{t_r^2\left[1 - \left(1 + \alpha/t_r\right) e^{-\alpha/t_r}\right]}. \quad (25)$$

The model parameters' MLL estimator is produced by numerically solving equation (18) after assigning it to zero. In the case of a complete sample, we acquire the MLL estimators of the model parameters for $r\ n$.

3.2. Simulation Results. A simulation is used to evaluate the estimators' behavior considering a set of parameter choices. Mean square error (℘), bias (I), lower limit ([) of the COIs, upper bound (H) of the COIs, and average length (ℶ) of 90% and 95% are among the metrics computed. All numerical calculations are made using the R programming (R 4.1.1). The following algorithm are used:

(i) On aggregate, the ILBE distribution produces 1000 random samples with sizes of n 100, 200, and 300. (ii) Values for a few parameters are α 1.2 and α 1.5.

(iii) There are three degrees of censorship: r 60%, 80% (TIIC), and 100% (complete sample).

(iv) ℘, I, [, H, and ℶ of estimates are computed.

Tables 1 and 2 include the numerical findings for the complete and TIIC measurements, respectively.

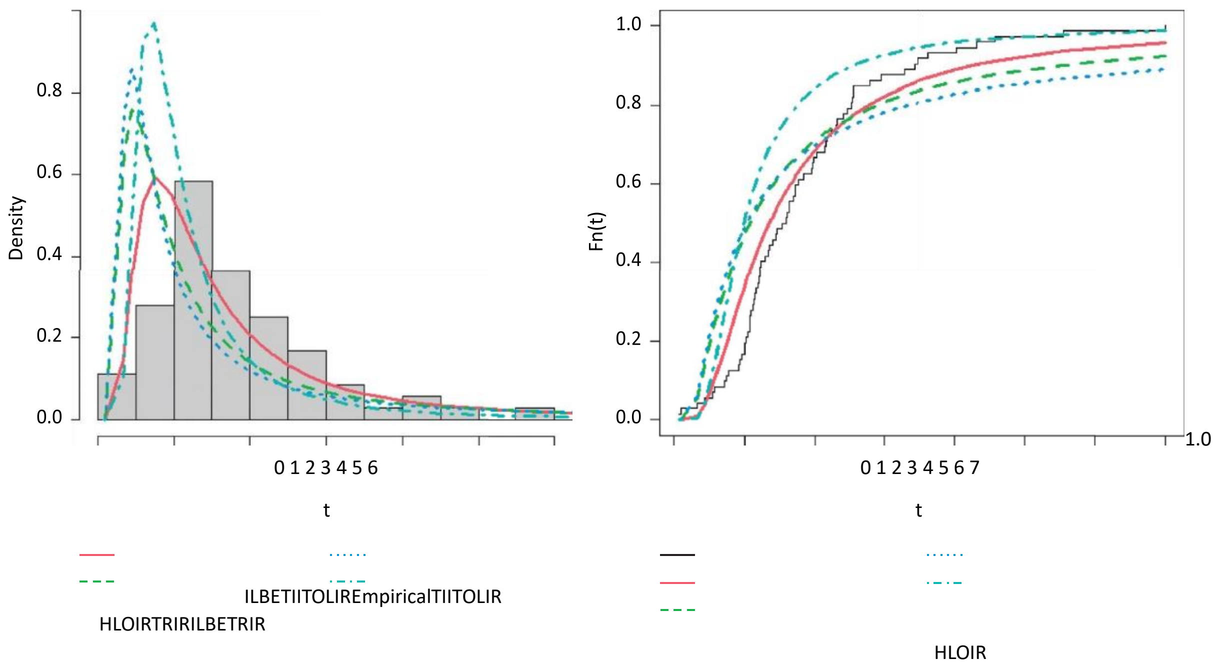


Figure 2: Fitted PDFs and CDFs of comparison models for the first dataset.

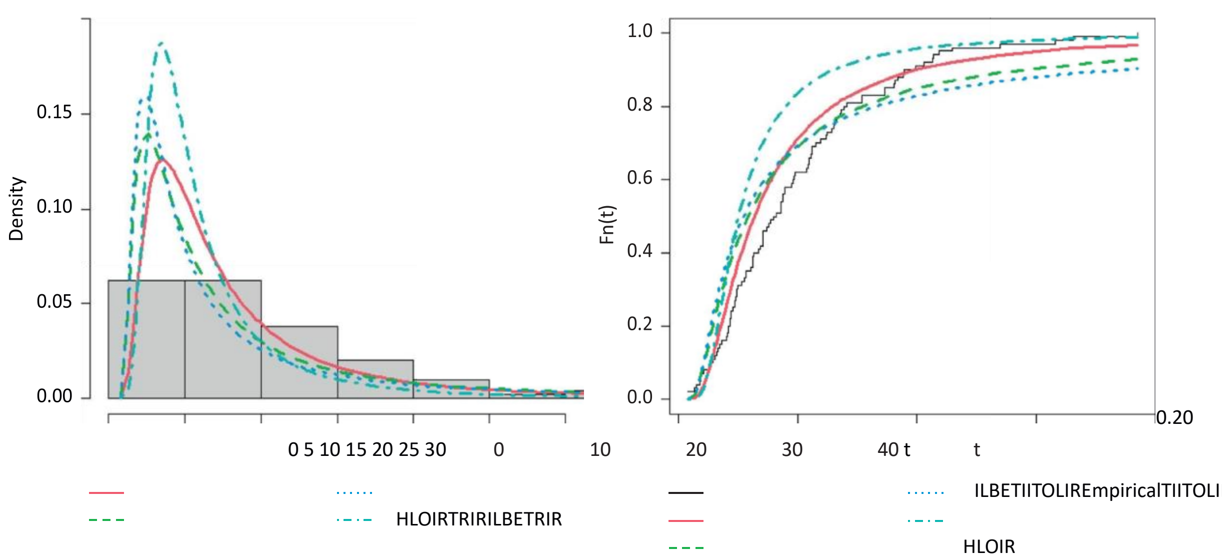


Figure 3: Fitted PDFs and CDFs of comparison models for the second dataset.

From these tables, we conclude the following: the ILBE model with known distributions such as the HLOIR [19], TIITOLIR [20], and TRIR [21] distributions,

(i) As the sample size grows, ℘, I, and ב of all estithe ILBE model performs better. The PDFs of competitive mates decrease. models are (ii) ℘, I, and ב of all estimates decrease as *r* decreases.

(iii) ב of the COIs increases as the confidence levels $4\lambda\alpha^2 t^{-3}\exp(-(\alpha/t)^2)(1-\exp(-(\alpha/t)^2))^{\lambda-1}$ increase from 90% to 95%.

$f_{HLOIR}(t)$ ———— 2 λ 2 ,

(1 + (1 − exp(− (α/t)))

4. Applications to Real Data

$f_{TIITOLIR}(t)\ 4\vartheta\alpha t^{2-3} \exp(-2(\alpha/t)^2)(1-\exp(-2(\alpha/t)^2))^{2\vartheta-1}$,

In this part, we demonstrate the ILBE model's adaptability (26) by examining three real-world datasets. Comparing the fit of

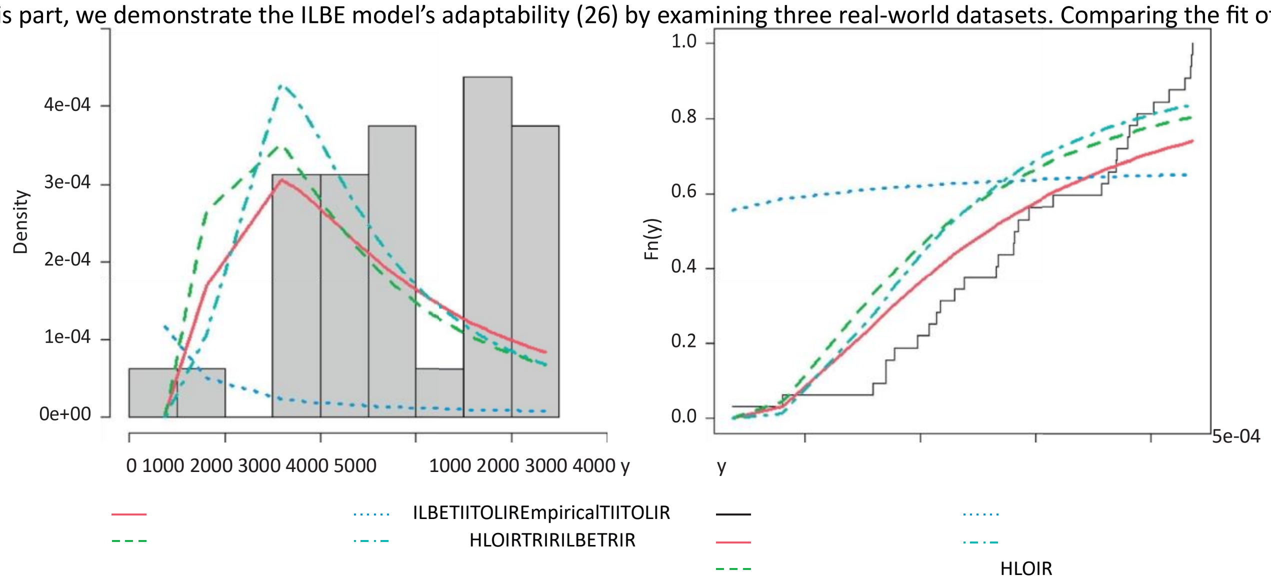


Figure 4: Fitted PDFs and CDFs of comparison models for the third dataset.

Table 3: Numerical results of MLE, SE, ∧**1**, ∧**2**, ∧**3**, ∧**4**, and ∧**5** for the first dataset.

	MLEs and SE		∧1	∧2	∧4	∧5	∧3
(α)	2.272 (0.189)		224.111	226.111	225.969	227.018	226.168
(α, λ)	0.436 (0.05)	0.579 (0.07)	260.586	264.586	264.301	266.399	264.76
(α, λ)	0.325 (0.036)	0.404 (0.058)	280.492	284.492	284.207	286.305	284.666
(α, λ)	0.352 (0.426)	−0.942 (0.351)	280.538	284.538	284.253	286.351	284.712

Table 4: Numerical results of MLE, SE, ^**1**, ^**2**, ^**3**, ^**4**, and ^**5** for the second dataset.

Model	MLEs and SE		^1	^2	^4	^5	^3
ILBE (α)	10.696 (0.7563)		664.794	666.794	666.794	667.848	666.834
HLIR (α, λ)	2.404 (0.226)	0.589 (0.06)	680.806	684.806	684.806	686.915	684.93
TIITLIR (α, λ)	1.824 (0.162)	0.43 (0.051)	700.214	704.214	704.214	706.323	704.338
TIR (α, λ)	9.978 (1.136)	− 0.812 (0.085)	720.665	724.665	724.665	726.774	724.706

Table 5: Numerical results of MLE, SE, ^**1**, ^**2**, ^**3**, ^**4**, and ^**5** for the third dataset.

Model	MLEs and S.E		^1	^2	^4	^5	^3
ILBE (α)	4326 (540.7191)		567.141	569.141	568.647	569.627	569.275
HLIR (α, λ)	1237 (184.326)	0.866 (0.172)	575.242	579.242	578.252	580.214	579.656
TIITLIR (α, λ)	0.069 (0.041)	0.049 (0.0091)	716.204	720.204	719.214	721.176	720.618
TIR (α, λ)	1821000 (346600)	− 0.859 (0.126)	575.303	579.303	578.313	580.275	579.717

$$f_{TRIR}(t)\ 2\vartheta t^{-3} \exp\left(-\frac{\vartheta}{t^2}\right)\left[1+\lambda-2\lambda \exp\left(-\frac{\vartheta}{t^2}\right)\right].\ t \quad t \tag{27}$$

In order to make a comparison between various models, some information criteria (INC) like maximized likelihood (?**1**), Akaike INC (?**2**), consistent Akaike INC (?**3**), Bayesian INC (?**4**), and Hannan–Quinn INC (?**5**) are used. According to the given data, the optimal model is one with the lowest value of ?**1**, ?**2**, ?**3**, ?**4**, and ?**5**.

The first dataset [22]: it describes 72 guinea pigs infected with highly pathogenic tubercle bacilli and their survival periods (in days).

The second dataset: acquired and documented in [23], the dataset comprises the waiting times (in minutes) of 100 bank clients.

The third dataset [24]: it offers 32 observations on the failure time for vertical boring machines.

Figures 2–4 indicate the fitted PDFs, fitted CDFs of the ILBE distribution, and those of the comparison models (HLOIR, TIITOLIR, and TRIR) for the three datasets.

It can be observed from Figures 2–4 that the ILBE distribution exhibits good matches, attesting its applicability for the three datasets.

Tables 3–5 show the ML estimates (MLEs) and standard errors (SEs) for the ILBE model when compared to various known distributions such like HLOIR, TIITOLIR, and TRIR. They also include the relevant measures of fit statistic.

Furthermore, Tables 3–5 show that the ILBE distribution is the best match among the other models for the three datasets, since the ILBE distribution has the lowest values of the suggested metrics.

5. Conclusions

This paper developed a new one-parameter lifetime distribution, named as inverse length-biased exponential distribution. The new model is quite flexible in nature and can acquire a variety of shapes of density and hazard rate functions. MOs, PRWMOs, inverse MOs, incomplete MOs, MRLS, and MINT are all explored as key characteristics of the new distribution. In both complete and censored samples, the maximum likelihood methodology is developed to calculate the parameters of the new distribution. To investigate the conduct of estimations, a simulation analysis is discussed. Three real-world examples show that the inverse length exponential distribution gives a pretty good fit and may be used as a competitive model to fit real-world data. It is hoped that this distribution would be helpful to scholars in a variety of disciplines. In the future, we plan to use the new proposed model to study the statistical inference of it under different censored schemes, using various methods of estimation to assess the performance of its parameters. Also, researchers can extend and generalized it because this model is very simple and has more flexibility to fitting more datasets.

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