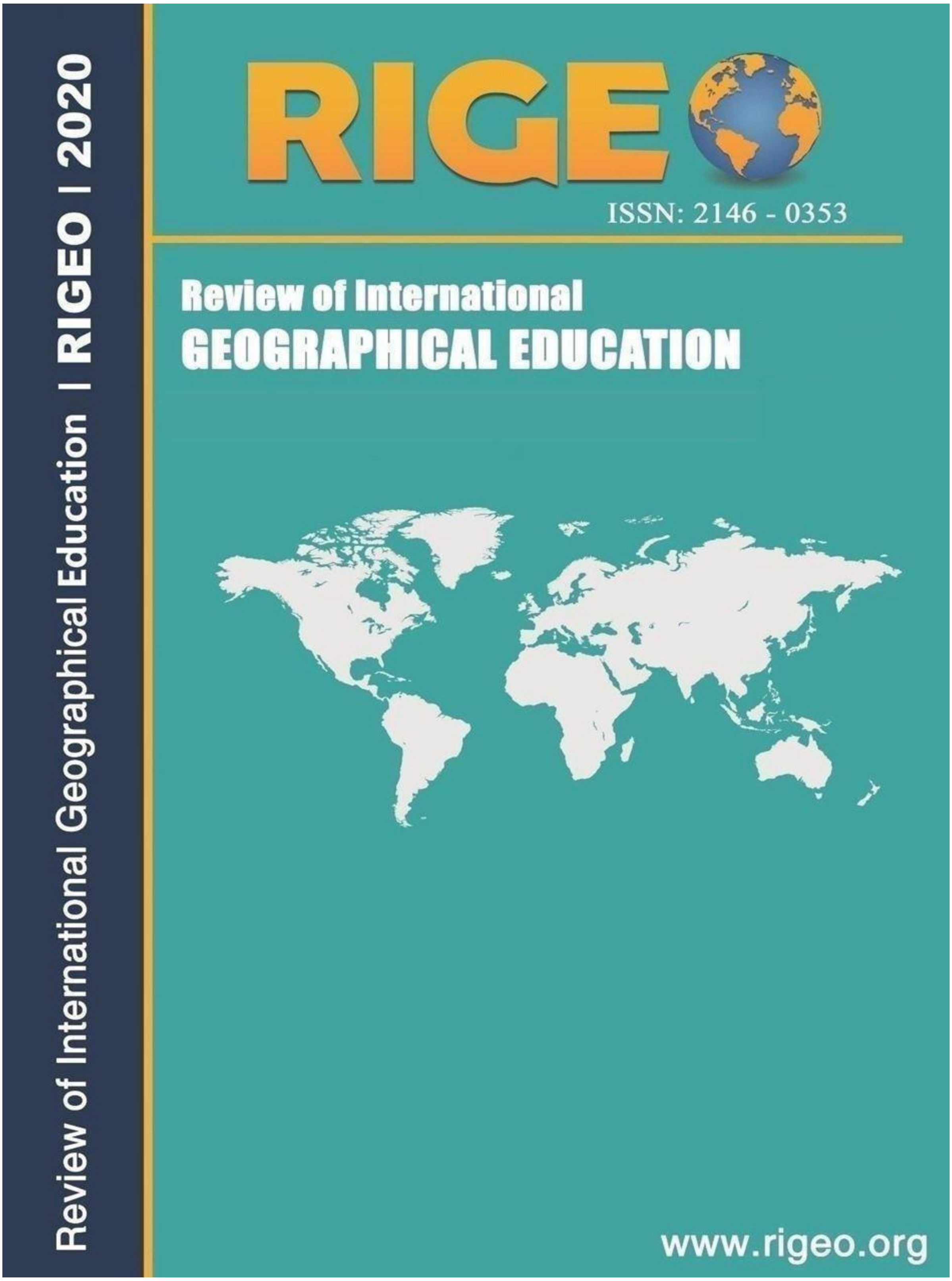
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Numerical Range of the Parallel Sum of Two Orthogonal Projections Characterized Geometrically

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1. Introduction

Let $\mathcal{H}$ be a complex separable Hilbert space with inner product $\langle\cdot, \cdot\rangle$ and $\mathcal{B}(\mathcal{H})$ be the algebra of bounded linear operators on $\mathcal{H}$. The numerical range $W(T)$ of an operator $T \in \mathcal{B}(\mathcal{H})$ is defined as

$$W(T) = \{\langle Tx, x\rangle : x \in \mathcal{H}, \|x\| = 1\}. \quad (1)$$

It is known that $W(T)$ is a nonempty bounded convex set in the complex plane $\mathbb{C}$ and its closure, denoted by $\overline{W(T)}$, always contains the spectrum $\sigma(T)$ of T (see [1, 2]). In addition, for $T_1, T_2 \in \mathcal{B}(\mathcal{H})$, we have $W(T_1 \oplus T_2) = \operatorname{conv}(W(T_1) \cup W(T_2))$, where $\operatorname{conv} S$ stands for the convex hull of the set S. For references on the numerical range and its generalizations, see, for instance, [3–8].

This paper arose from an attempt to gain a geometric characterization of the numerical range of parallel sum with a view of operator block. In what follows we always suppose $A, B \in \mathcal{B}(\mathcal{H})$ and $A + B$ has closed range. The parallel sum of A and B is defined as

$$A : B = A(A + B)^{\dagger}B, \quad (2)$$

where $T^{\dagger}$ is the Moore–Penrose generalized inverse of T (see [9, 10]). The study of parallel sum is motivated by the fact that if A and B are impedance operators of resistive n-port electrical networks, then $A : B$ is the impedance operator of the parallel connection [11]. Several authors, in particular Anderson and Trapp [11], Anderson and Duffin [12], Ando [13], and Wang et al. [10], extended this result and estab- lished many diferent equivalent definitions and properties on parallel sum (see also [9, 10]). Recently, Klaja [14] applied Halmos' two projections theorem to describe the numerical range of a product of two orthogonal projections P and Q. He showed that the closure of its numerical range is equal to a closed convex hull of some ellipses parametrized by points in the spectrum. In [8], Wang et al. also used Halmos' two projections theorem to study the containment region of the numerical range of the product of a pair of positive con- tractions. Zhang and Yu [15] described the numerical range of the operator P QP. Motivated by these, we consider the numerical range of the parallel sum P : PQ for orthogonal projections P and Q. The investigation uses in an essential way Halmos' two projections theorem, which is introduced as follows.

Let P and Q be two orthogonal projections on $\mathcal{H}$. Thus, $P = P^2 = P^*$ and $Q = Q^2 = Q^*$. The ranges of P and Q are denoted by $\mathcal{L}$ and $\mathcal{N}$, respectively. According to Halmos' two projections theorem (see [16] and consult [17] for the history and more on the subject), there is a representation of $\mathcal{H}$ as an orthogonal sum:

H � $(L \cap N) \oplus (L \cap N^{\perp} \oplus (L^{\perp} \cap N \oplus (L^{\perp} \cap N^{\perp} \oplus H,$

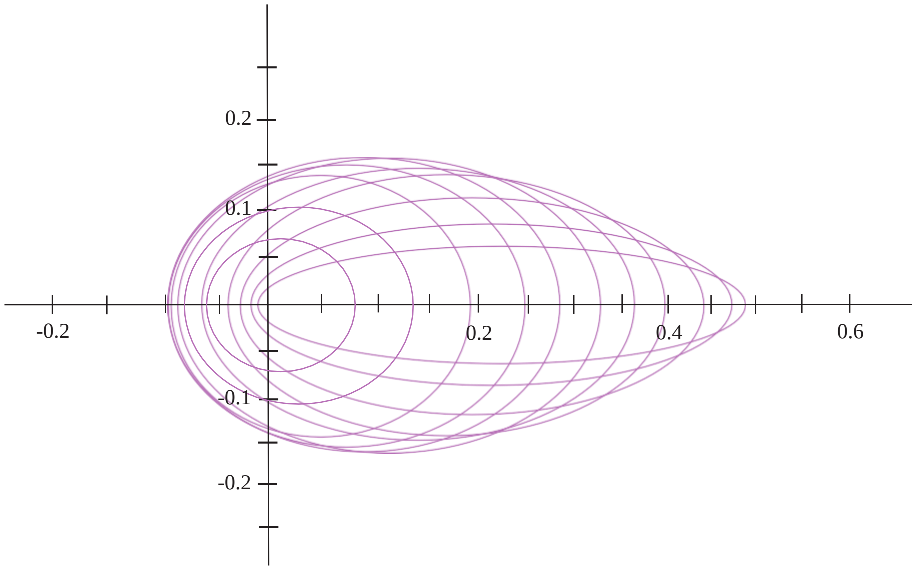

Figure 1: Ellipse $E(\lambda)$ for λ � 0.02, 0.05, 0.1, 0.15 . . . , 0.45, 0.48, 0.5.

Since $E(\lambda)$ is symmetric about y �0, only $\alpha \in 0,[\pi$]needs to be considered, and the proof will be divided intotwo cases.

Case One. Suppose that $\cos(\alpha) \neq 0$. It follows from

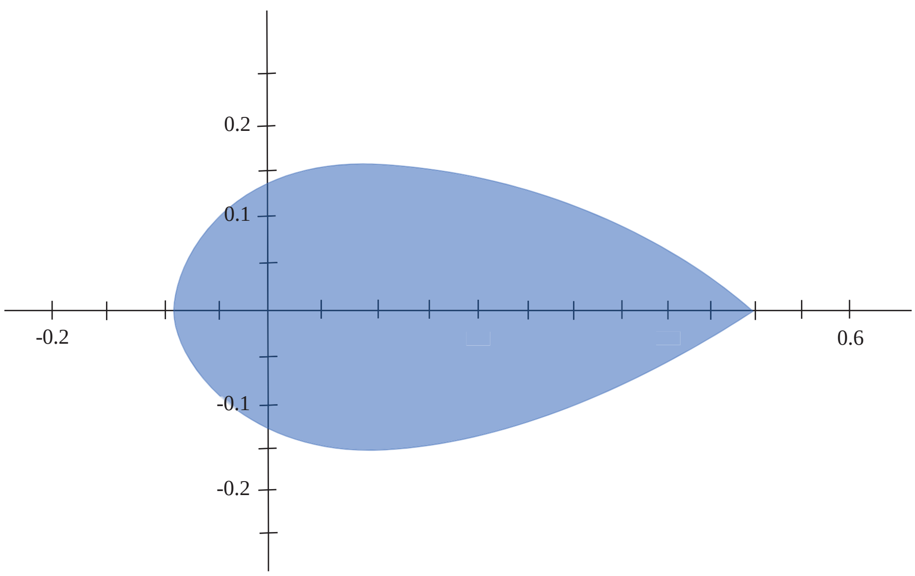

Figure 2: $\overline{\text{conv} \cup_{\lambda\in[0,1/2]} E(\lambda)}$.

mpleted. □ *Proof of Theorem 1.* From Lemmas 6 an

The proof is co

Then, we can prove Theorems 1 and 2.

1

( )( ) � λ 2

$\lambda \cos(\alpha) +$

\- $\lambda^2 \sin^2(\alpha)$ $\rho_{E(\lambda)}(\alpha)$.

$\in \sigma(P:PQ)$

It follows from Lemma 6 that

$$\overline{W(P: PQ)} \text{ � } \overline{\text{conv} \cup_{\lambda \in \sigma(P:PQ)} E(\lambda)}. \quad (43)$$

The proof is completed. □

Proof of Theorem 2. From the matrix form in (2), we have

$$P: PQ \sim \frac{1}{2} I \oplus 0 \oplus 0 \oplus 0 \oplus \frac{I + 3\cos^2(T)}{I + 3\cos^2(T)} \quad (44)$$

$\cos^2(T) I + \cos^2(T)$

$\cos(T)\sin(T) I + \cos^2(T)$

Suppose H into two cases. { }

� {0}. The following proof will be divided

Suppose $H \neq 0$. The following proof will be dividedinto two cases.

(1) If $L \cap N \neq \{0\}$, then $W(P: PQ)$ � $W(1/2I \oplus 0 \oplus 0 \oplus 0)$ � conv{{0} ∪ {1/2}} � [0, 1/2]. In this case, $\lambda \in \sigma(P: PQ)$ � {0,1/2} and $E(0)$ � {0}, $E(1/2)$ � [0, 1/2]. Thus, $W(P: PQ)$ � [0, 1/2] � conv{$E(0)$ ∪ $E(1/2)$} � conv ∪$_{\lambda \in \sigma(P:PQ)} E(\lambda)$.

(3) If $L \cap N$ � {0}, we have $P: PQ$ � 0 on the space $(L \cap N) \oplus H$. Thus, the closure of the numerical range of $P: PQ$ on the space $(L \cap N) \oplus H$ is

{0} ⊂ $E(\lambda)$

(2) If $L \cap N$ � {0}, we have $W(P: PQ)$ � {0} � $E(0)$. for all $\lambda \in [0, 1/2]$, we can have

$\overline{W(P: PQ)}$ � $\overline{\text{conv} \cup_{\lambda \in \sigma(P:PQ)} E(\lambda)}$ on the space $(L \cap N) \oplus H$.

(4) If $L \cap N \neq \{0\}$, we have $P: PQ$ � $1/2I$ on the space $(L \cap N) \oplus H$. Thus, the closure of the numerical range of $P: \underline{PQ}$

on the space $(\mathsf{L} \cap \mathsf{N}) \oplus \mathsf{H}$ is conv{1/2} ∪ conv $\cup_{\lambda\in\sigma(P:PQ)}\mathsf{E}(\lambda)$}. As {0} ⊂ E

(λ) for all $\lambda \in [0, 1/2]$ and

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combination of 0 and 1/2 is contained in the closure of the numerical range of $P: PQ$. Thus, the closure of the numerical range of P: PQ on the space

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$(\mathsf{L} \cap \mathsf{N}) \oplus \mathsf{H}$ is conv{[0, 1/2]

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the space $(\mathsf{L} \cap \mathsf{N}) \oplus \mathsf{H}$. The proof is completed. □

Corollary 9. *Let P and Q be orthogonal projections. Then, for $\lambda \in \sigma(P: PQ)$, we can get*

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$\overline{W(P: PQ)}$ � conv

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In particular, we have $W(P: PQ)$ � conv $\cup_{\lambda\in[0,1/2]}\mathsf{E}(\lambda)$ when $\sigma(P: PQ)$ � [0, 1/2], as shownin Figure 2 [21–24].

Acknowledgments

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